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Alfred MBAIRADJIM
J. SADEFO KAMDEM
Arnold F. SHAPIRO
Michel TERRAZA

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Capital asset pricing model with fuzzy returns and hypothesis testing

Alfred Mbairadjim M. Lameta
Université Montpellier I

J. Sadefo Kamdem *
LAMETA
Université Montpellier I

Arnold F. Shapiro
Smeal College of Business
Penn State University

M. Terraza
LAMETA-CNRS
Université Montpellier I

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Abstract

Over the last four decades, several estimation issues of the beta have been discussed extensively in many articles. An emerging consensus is that the betas are time-dependant and their estimates are impacted by the return interval and the length of the estimation period. These findings lead to the prominence of the practical implementation of the Capital Asset Pricing Model. Our goal in this paper is two-fold: After studying the impact of the return interval on the beta estimates, we analyze the sample size effects on the preceding estimation. Working in the framework of fuzzy set theory, we first associate the returns based on closing prices with the intraperiod volatility for the representation by the means of a fuzzy random variable in order to incorporate the effect of the interval period over which the returns are measured in the analysis. Next, we use these fuzzy returns to estimate the beta via fuzzy least square method in order to deal efficiently with outliers in returns, often caused by structural breaks and regime switches in the asset prices. A bootstrap test is carried out to investigate whether there is a linear relationship between the market portfolio fuzzy return and the given asset fuzzy return. Finally, the empirical results on French stocks suggest that our beta estimates seem to be more stable than the ordinary least square (OLS) estimates when the return intervals and the sample size change.

Key-words: CAPM ; Beta; Fuzzy least squares; Bootstrap hypothesis testing; Intraperiod volatility.

*Corresponding author. LAMETA Université de Montpellier I, UFR d’Economie Avenue Raymond DUGRAND - Site de Richter C.S. 79606 34960 MONTPELLIER CEDEX 2 France; Courriel: sadefo@lameta.univ-montp1.fr, jules.sadefo-kamdem@univ-montp1.fr
1 Introduction

By introducing the Capital Asset Pricing Model (CAPM), Sharpe (1963), Lintner (1965) proposed the first quantified approach of risk in quantitative finance. They determined the systematic risk or beta by covariance with the market. The beta is commonly computed by applying the standard market model estimated under ordinary least square (OLS).

As with most important scientific models, the CAPM has been subject to substantial criticism, the most famous of which addressed by Fama and French (1992). A large literature over several decades focused on the estimation issue of the CAPM, more precisely, of the systematic risk or beta. One of the first criticisms is the instability of its estimates across time. Some researchers such as Blume (1971, 1975), Ferson and Harvey (1991, 1993) and Ferson and Korkajczyk (1995), find that estimated betas exhibit statistically significant time variation. For this reason, these last three works suggest replacing the static CAPM by some forms of time-varying beta and of conditional CAPM. This approach allows out-performing the constant beta if the dynamic of the beta is captured with success. Some conditional versions of beta and a formal characterization of their persistence and predictability vis-à-vis their underlying components, was also introduced by Andersen et al. (2004). However, as shown by Ghyseys (1998), if the dynamic of the beta is misspecified, there is a real possibility of pricing errors potentially larger than with a constant beta assumption.

Another important estimation issue of the beta, extensively investigated by several authors, is the impact that the return interval has on the beta estimate. Brailsford and Josev (1997), Hawawini (1983), Handa et al. (1989, 1993), among others, report that different beta estimates can be obtained over the same period by changing the interval over which the return is calculated. Furthermore, Handa et al. (1993) empirically reject the CAPM when monthly returns are used but they accept the CAPM with yearly returns whereas Fama (1981, 1990) show that the power of macroeconomic variables in explaining the stock prices increased with time length. However, the early work of Levhari and Levy (1977) provide evidence that the beta estimates were biased if the analyst used a shorter time horizon than the relevant time horizon implicit in the decision making process of investors. For this reason, they suggest to use the relevant

Beta is typically defined as the slope parameter of the Sharpe’s market line which is given by

\[ R_{it} = \alpha_i + \beta_i R_{mt} + \epsilon_{it}, \]

where \( R_{it} \) and \( R_{mt} \) are respectively the realized returns on the asset \( i \) and on the market portfolio \( m \) over the return interval \( t \); \( \alpha_i \) is the constant term for the asset \( i \); the error term \( \epsilon_{it} \) is a gaussian random variable defined through the variance \((Var)\) and the covariance \((Cov)\) as follows

- zero-mean: \( E(\epsilon_{it}) = 0, \forall t, \forall i \),
- homoscedastic: \( Var(\epsilon_{it}) = \sigma_{\epsilon_i} \),
- mutually uncorrelated in the time: \( Cov(\epsilon_{it}, \epsilon_{it'}) = 0, \forall t \neq t' \),
- \( \epsilon_{it} \) are independent of market return \( R_{mt} \): \( E(\epsilon_{it} | R_{mt}) = 0, \forall t, i \).
time horizon in the decision making process in order to avoid biasing the beta estimate even if information is available in a higher frequency. A solution based on wavelet analysis, was proposed by Gençay et al. (2005). They decomposed the time series, measured at the highest possible frequency, into different time scales before investigating the beta behavior at different time horizons without losing information.

Financial assets closing prices generally contain noise because of the imperfections of the trading process. As noticed in Aït-Sahalia et al. (2010), these imperfections might be largely divided into three parts. The first represents the frictions inherent in the trading process: bid-ask bounces, discreteness of price changes and rounding, trades occurring on different markets or networks, etc. The second point concerns informational effects such as differences in trade sizes or informational content of price changes. The last point encompasses measurement or data recording errors. Thus, returns based on closing prices are variables measured with errors. It follows that the use of returns based on closing prices, tends to lead to inconsistent ordinary least square estimators in market line or multifactor models. Cragg (1994) demonstrated that the slope coefficients are biased toward zero and concluded that the measurement error "produces a bias of the opposite sign on the intercept coefficient when the average of the explanatory variables is positive".

In this paper, we study the CAPM which remains central to financial economics despite the numerous criticisms that it is subject to. We propose a new approach for estimating the systematic risk. The motivation behind our approach comes from two sources. First, we want to use a relevant return interval in the investor’s decision making process and to take into account the higher frequency data observed within the considered interval, in order to decrease the loss of information often caused by a huge discretization in time. Second, we aim to introduce an estimation method which is better able to deal with the impacts of the estimation period length (Kim (1993)), and that structural breaks and other regime switches (Bundt et al.(1992), Garcia and Ghysels (1998)) on the betas estimate. The basic assumption of our modeling approach is the representation of financial assets returns through a fuzzy random variable in order to capture different relevant information on the probability distribution of the intraperiod returns observed. The fuzzy representation of financial asset return has been explored in literature by many authors including among others, Tanaka and Guo (1999), Parra et al. (2001), Terol et al. (2006) and Yoshida (2009).

We study the fuzziness of returns over a period, as the intraperiod volatility. This modeling aims to deal with the impact of return intervals on the returns representation and to produce estimates less sensitive to sample size effects. We apply a fuzzy computation method to define the fuzzy returns and to fit the fuzzy market line (FML) as a fuzzy linear regression model.

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2Klepper and Leamer (1984) and Leamer (1987), among others, provide evidence of inconsistency of ordinary least square estimators in linear regression models with measurement errors in the regressors.

3A conceptual discussion on fuzzy random variable is given by Shapiro (2009)
Our fuzzy set-valued returns can be seen as a generalized form of the interval-valued one. Furthermore, the fuzzy linear regression model introduced by Tanaka et al. (1982) and refined by Diamond (1988), is more adequate for dealing with outliers, and small sample size, as it is in our case, than the linear regression model.

The remainder of this paper is organized as follows. Section 2 briefly reviews some basic concepts of fuzzy set theory. Section 3 describes the fuzzy presentation process of monthly returns. Section 4 introduces the fuzzy market line, its implications and the hypothesis testing approach of the beta estimates. Section 5 is an empirical study based on the French index, which compares our beta estimates to the OLS estimates and the wavelet estimates. Finally some conclusions are listed in Section 6. In addition, for a better understanding of the paper, a presentation of the bootstrap test method applied in the article and a brief review of wavelet analysis are given in Appendix A and B respectively.

2 Preliminaries

Before proceeding to formal presentation of the capital asset pricing model with fuzzy returns, we first briefly review three of the basic concepts of fuzzy theory; namely fuzzy sets, fuzzy numbers and fuzzy random variables. Readers familiar with these topics can skip this section, and those interested in a detailed presentation of fuzzy theory, may see Zimmermann(2001).

2.1 Fuzzy sets and fuzzy numbers

Let $X$ be a crisp set. A fuzzy subset $A$ of $X$ is defined by its membership function $\mu_A : X \rightarrow [0, 1]$ which associates each element $x$ of $X$ with its membership degree $\mu_A(x)$ (Zadeh (1965)). The degree of membership of an element $x$ to a fuzzy set $A$ is equal to 0 (respectively 1) if we want to express with certainty that $x$ does not belong (respectively belongs) to $A$.

The crisp set of elements that belong to the fuzzy set $A$ at least to the degree $\alpha$ is called the $\alpha$-cut or $\alpha$-level set and defined by:

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}$$  \hspace{1cm} (1)

$A_0$ is the closure of the support of $A$.

Fuzzy numbers have some properties, examples of which are the notions of "around ten percent" and "extremely low". Dubois and Prade (1980, p.26) characterizes the fuzzy numbers as follows:

4The closure of the support of $A$ is the smallest closed interval containing the support of $A$ (Shapiro (2009))
5The support of $A$ is the set of all $x$ such that $\mu_A(x) > 0$. (Shapiro (2009))
Definition 2.1 A fuzzy subset \( A \) of \( \mathbb{R} \) with membership \( \mu_A : \mathbb{R} \to [0, 1] \) is called fuzzy number if

1. \( A \) is normal, i.e. \( \exists x_0 \in \mathbb{R} | \mu_A(x_0) = 1 \);
2. \( A \) is fuzzy convex, i.e.
   \[ \forall x_1, x_2 \in \mathbb{R} \mid \mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}, \forall \lambda \in [0, 1]; \]
3. \( \mu_A \) is upper semi-continuous\(^6\);
4. \( \text{supp}(A) \) is bounded.

Definition 2.2 (Zimmermann(1996, p. 64)) A LR-fuzzy number, denoted by \( \tilde{A} = \langle l, c, r \rangle_{LR} \), where \( c \in \mathbb{R}^+ \) is called central value, and \( l \in \mathbb{R}^+ \) and \( r \in \mathbb{R}^+ \) is the left and the right spread, respectively, is characterized by a membership function of the form

\[
\mu_A(x) = \begin{cases} 
    L\left(\frac{x-l}{c-l}\right) & \text{if } c-l \leq x \leq c, \\
    R\left(\frac{x-r}{r}\right) & \text{if } r+c \geq x \geq c, \\
    0 & \text{else}.
\end{cases}
\]  

(2)

\( L : \mathbb{R}^+ \to [0, 1], \ R : \mathbb{R}^+ \to [0, 1] \) are fixed left-continuous and non-increasing functions with \( R(0) = L(0) = 1 \) and \( R(1) = L(1) = 0 \). \( L \) and \( R \) are called the left and the right shape functions respectively. If right and left spreads are equal and \( L := R \), the LR-fuzzy number is said to be a symmetric fuzzy number and denoted \( \tilde{A} = (c, \Delta) \). \( \Delta \) is the spread equal to \( l = r \).

Without loss of generality, we limit the present study to triangular fuzzy\(^7\) characterized by the shape functions \( R(x) := L(x) := \max\{1 - x, 0\} \).

Using Zadeh’s extension principle (Zadeh (1965)), which is a rule providing a general method to extend a function \( f : \mathbb{R}^k \to \mathbb{R} \) to the set of fuzzy numbers, we can define binary operator such as addition, subtraction, multiplication for two fuzzy numbers. When \( k = 2 \), this method defines the membership function of the result as follows

\[
\mu_{\tilde{A}_1 \circ \tilde{A}_2}(z) = \sup_{(x_1, x_2) \in \tilde{A}_1 \times \tilde{A}_2} \{\min\{\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2)\} \mid x_1 \circ x_2 = z\}
\]  

(3)

where \( \circ \) is the binary operator.

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\(^6\)Semi-continuity is a weak form of continuity. Intuitively, a function \( f \) is called upper semi-continuous at point \( x_0 \) if the function’s values for arguments near \( x_0 \) are either close to \( f(x_0) \) or less than \( f(x_0) \).

\(^7\)This assumption is also made in many articles such as Koissi and Shapiro(2006), Andrés-Sánchez (2007) and Berry-Stolze et al. (2010), among others.
Several distances between fuzzy numbers are defined in the literature for very specific purposes. The metric \( \delta_2 \) introduced by Bertoluzza et al. (1995) in the case of a one-dimensional convex set, is one of the more commonly used for least square problems (Näther (2006), González-Rodríguez et al. (2009)). For two symmetric LR-fuzzy numbers \( A_i = (a_i, \Delta_i) \), \( i = 1, 2 \), with shape function \( L : \mathbb{R}^+ \to [0,1] \), it is defined by:

\[
\delta_2(A_1, A_2) = \sqrt{(a_1 - a_2)^2 + l(\Delta_1 - \Delta_2)^2}
\]

where

\[
l = \int_0^1 \left( L^{-1}(\alpha) \right)^2 d(\alpha)
\]

2.2 Fuzzy random variables

Different approaches of the concept of fuzzy random variables have been developed in the literature since the 70’s. The most often cited being introduced by Kwakernaak (1978) and enhanced by Kruse and Meyer (1987), and the one by Puri and Ralescu (1986). An extensive discussion on these two approaches is given by Shapiro (2009). For the purpose of this study, we adopt the concept of FRVs of Puri and Ralescu (1986).

Let \( \mathcal{F}_c(\mathbb{R}) \) denote the set of all normal convex fuzzy subsets\(^8\) of \( \mathbb{R} \) and \( (\Omega, \mathcal{A}, P) \) a probability space.

More precisely, Puri and Ralescu (1986) have defined a FRV as follows

**Definition 2.3** The mapping \( X : \Omega \to \mathcal{F}_c(\mathbb{R}) \) is said to be a FRV on \( \mathbb{R} \) if for any \( \alpha \in [0,1] \), the \( \alpha \)-cut is a convex compact random set\(^9\).

Puri and Ralescu (1986) brought an expectation operator called the Aumann-Expectation and denoted \( E^A \) for FRVs. Its construction is based on the Aumann’s (1965) study on integrals of interval-valued functions. For a symmetric LR-fuzzy random variable \( \tilde{A} = \langle a, \Delta \rangle \), the Aumann-expectation \( E^A \) is defined by (Körner (1997)):

\[
E^A(\tilde{A}) = \langle E(a), E(\Delta) \rangle
\]

Definition \(^6\) shows that the Aumann-expectation is a linear operator as the expectation operator for real random variables.

Following the Puri and Ralescu’s approach of a FRV and based on the definition\(^6\) Körner (1997)

\(^8\) A fuzzy set \( \tilde{A} \) is called a normal convex fuzzy subset of \( \mathbb{R} \) if \( \tilde{A} \) is normal, the \( \alpha \)-cuts of \( \tilde{A} \) are convex and compact and the support of \( \tilde{A} \) is compact. (Körner(1997))

\(^9\) Where \( \Omega \) is the set of all possible outcomes described by the probability space, \( \mathcal{A} \) is \( \sigma \)-fields of subsets of \( \Omega \), and the function \( P \) defined on \( \mathcal{A} \) is a probability measure.

\(^{10}\) A convex compact random set is a Borel-measurable mappings with the Borel \( \sigma \)-field generated by the topology associated with the Hausdorff metric on \( \mathcal{F}_c(\mathbb{R}) \). (Gil et al. (2006))
introduced a real-valued variance characterized by the Fréchet principle¹¹ and covariance. For LR-fuzzy numbers \( \tilde{A}_i = \langle a_i, \Delta_i \rangle, i = 1, 2 \), Körner (1997) describes the variance and covariance operators as follow:

\[
\text{Cov}^A (\tilde{A}_1, \tilde{A}_2) = \text{Cov}[a_1, a_2] + l \times \text{Cov}(\Delta_1, \Delta_2),
\]

\[
\text{Var}^A (\tilde{A}) = \text{Var}(a) + l \times \text{Var}(\Delta).
\]

where \( l \) is defined as in (5). For triangular fuzzy numbers, \( l = 1/3 \).

3 Fuzzy representation of returns

Fuzzy random variables (FRV) were introduced and defined by Kwakarnaak (1978), Puri and Ralescu (1986) as a well-formalized model for fuzzy set-valued random elements. Since these definitions, numerous studies in probability theory have been developed to analyze the properties of this new class of random variables. For the last three decades, we can cite those related to the formalization of the measurability, to the laws of large numbers which strengthen the suitability of the fuzzy mean and to the hypothesis testing (See Gil et al. (2006) for an overview of these developments). Despite the existence of this complete mathematical analysis framework, the application of these theoretical results is still quite limited because of difficulties related to observing and measuring FRVs in practice. Hence, the necessity of building methods to provide fuzzy representations of observations, which are often crisp. A solution was proposed by Gonzalez-Rodriguez et al. (2006) who introduced a family of fuzzy representation of random variables. Each of the representations transforms a crisp random variable into a fuzzy random variable whose mean captures different relevant information on the probabilistic distribution of the original real-valued random variable. However the application of this method requires \( \text{à priori} \) assumptions about the distribution of the real variable and about the shape of the membership function of the fuzzy random variable. This two-fold assumption may lead to a significant bias of information.

Furthermore, koissi and Shapiro(2006, p.291)¹² specified that a crisp data can be fuzzified by adding a number \( \pm \Delta \) to each value, where \( \Delta \) is chosen small compared to the center value. Following this approach, in what follows, we fuzzify the observed return by choosing \( \Delta \) as a measure of the realized volatility within the period over which the return is calculated. Since

\[\begin{align*}
\text{Fréchet (1948)} & \text{ the expectation operator } \mathbb{E}^{(d)}[Z] \text{ and the variance } \mathbb{V}ar^{(d)}[Z] \text{ for a random variable } Z \text{ in a metric space } (M, d) \text{ as } \\
\mathbb{E}[d^2(Z, \mathbb{E}^{(d)}Z)] &= \inf_{x \in M} \mathbb{E}[d^2(Z, x)], \\
\mathbb{V}ar^{(d)}[Z] &= \mathbb{E}[d^2(Z, \mathbb{E}^{(d)}Z)].
\end{align*}\]

¹¹ Fréchet (1948) also recall from other authors, that the choice \( \Delta \) might be arbitrary (Chang and Ayyub (2001,p.192), randomly generated (Diamond (1988, p.152) or resulting from fuzzy regression Chang and Ayyub (2001)).
the realized volatility is not only a measure of deviation but also depends on the time scale (length of the period) over which returns are calculated our fuzzy return aims at reflecting the interval effect and the imprecision induced the market microstructure noise.

Making an assumption on the left and right shape functions, we propose to fuzzify the return over a given period using past prices time series at higher frequency, as follows

The asset price time series is initially partitioned into sub-groups according to periods (months in our case). For each month, successively observed daily returns are calculated and we consider their corresponding empirical probabilistic distribution. Then the first two moments (expected value and variance) and the return over the month, based on the first and last values of the price time series, are computed. The monthly return is then represented as a symmetric LR-fuzzy number with a central value equal to the return over the month and the spread is the scaled standard deviation.

The fuzzy representation process can be summarized as in the following procedure.

We denote by \( t \) the sub-period \([t, t+1]\)

**Procedure 3.1.**

1. **Step 1:** Partition the price time series in sub-groups \( P_t = \{P_t+i/n, i = 0, \ldots, n\} \) with size \( n+1 \) each one corresponding to a period \( t \). The sample size \((n+1) \geq 2\) has to be sufficiently large.

   For each period \( t \)

2. **Step 2:** Compute the return over the period \( R_t = \frac{P_{t+1}-P_t}{P_t} \)

3. **Step 3:** Compute the returns within the period \( R_{t,i} = \frac{P_{t+1(i+1)/n}-P_{t+i/n}}{P_{t+i/n}}, \ i \in \{0, \ldots, n-1\} \)

4. **Step 4:** Estimate empirically the variance \( \hat{\sigma}_t^2 \) of \( R_{t,i} \) as

   \[
   \hat{\sigma}_t^2 = \frac{1}{n} \sum_{i=0}^{n-1} (R_{t,i} - \hat{\mu}_t)^2
   \]

   where

   \[
   \hat{\mu}_t = \frac{1}{n} \sum_{i=0}^{n-1} R_{t,i}
   \]

5. **Step 5:** Scale the intraperiod volatility by \( \Delta_t = \sqrt{n}\hat{\sigma}_t \)

6. **Step 6:** Fit the membership function of the symmetric LR-fuzzy return with central value \( R_t \) and spread \( \Delta_t \)

---

\(^{13}\)This behavior is one of the stylized facts of assets returns discussed by Cont (2000).

\(^{14}\)In this study, we use daily closing prices time series which are partitioned into monthly periods, hence \( n = 20 \).
The spread\textsuperscript{15} of the fuzzy return over the period $t$ is hence estimated as the realized volatility over this period. This volatility is computed using standard scaling practice\textsuperscript{16} which consists in multiplying the realized volatility and square root of the number of the observed returns within the period. This scaling method\textsuperscript{17} also known as the square-root-of-time rule, requires the assumption that the returns are independent and identically distributed. The fuzzy return then allows us to associate the closing prices-based returns to some other relevant information relative to their probabilistic distributions.

Finally, we have the following statement

**Proposition 3.2** If returns successively observed are supposed to be real random variables, the symmetric LR-fuzzy set constructed by the Procedure 3.1 is a fuzzy random variable as described in the definition 2.3.

4 The fuzzy market line (FML)

The numerous estimation issues of the systematic risk of the OLS method listed in the previous section motivate us to introduce the following new estimation method for the beta. Our aim is to analyze the effect of the return interval and the impact of the estimation period length on the beta estimates. The fuzzy representation of a period incorporates the interval over which the return is calculated by the mean of the scaled volatility. Regimes switches or structural breaks in the price process likely imply an outlier in the return time series. When the sample size increases by taking into consideration this kind of observation, the beta estimate from the OLS method changes, especially when the data set is too small. In order to deal efficiently with these limitations, we define the one-factor market model as a fuzzy linear regression model. The slope parameter of this model defines our systematic risk estimate.

4.1 Assumptions and model specification

For the CAPM with fuzzy returns, we adopt the assumption commonly used in asset pricing theory, in which all investors hold their wealth in the form of financial assets. We consider a single-period securities market economy and we assume that there exist one riskless asset and $n$ risky assets at time $t$. Except that the assumption of a normal distribution of return is replaced by the assumption that the returns are LR-fuzzy random variables, other assumptions are the same as in the original CAPM, including that capital market is completely competitive and frictionless, capital market clearing, riskless borrowing, and lending are allowed. The LR-fuzzy random return vector is denoted $\tilde{R} = (\tilde{R}_1, ..., \tilde{R}_n)$ and the riskless return is $R_f$. We assume

\textsuperscript{15}The term spread is used here in the fuzzy set theory framework. It is not related to the theoretically valid spread associated with the asset’s return.

\textsuperscript{16}Remark that the spread which is a measure of possible (not just probable) values of the return, is estimated by the scaling practice derived using a stochastic model. However, the appropriateness of stochastic methods for the fuzzification of crisp data, was argued by some authors such as Dubois et al. (2004), Dubois (2006).

\textsuperscript{17}A discussion on this scaling method is given by Danielsson and Zigrand (2005)
that $R_f$ is a fix positive real number.

For a risky asset $i$, we consider the Auman’s fuzzy expected return $E^A \left( \tilde{R}_i \right)$, its underlying standard deviation $\sigma^A_i$ and all correlations $\sigma^A_{ij}$ for the $n$ risky assets ($i,j = 1,...,n$).

Under these assumptions, if we denote $\tilde{R}_m$ as the fuzzy return of the market portfolio, we define the fuzzy market line by the following fuzzy linear regression

$$\tilde{R}_i = \tilde{\alpha}_i \oplus \beta \tilde{R}_m \oplus \tilde{\epsilon}_i,$$

where $\beta$ is an element of the real line ($\mathbb{R}$), $\tilde{\alpha}_i$ belongs to set of fuzzy numbers ($\mathcal{F}(\mathbb{R})$) and $\tilde{\epsilon}_i$ is a LR-fuzzy random variable, specified as follows:

$$E^A(\tilde{\epsilon}_i) = 1\{0\}, \forall i = 1,...,n;$$
$$Cov^A(\tilde{\epsilon}_i, \tilde{R}_m) = 0, \forall i = 1,...,n;$$
$$Cov^A(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0, \forall i,j = 1,...,n.$$

$1\{0\}$ denotes for the indicator function of a classical set. Note that the $Cov^A$ is defined via the Auman expectation.

The specification $\tilde{R}_i = \tilde{\alpha}_i \oplus \beta \tilde{R}_m \oplus \tilde{\epsilon}_i$ for the fuzzy market is also possible. However, the product $\beta \tilde{R}_m$ of two LR-fuzzy numbers does not always provide a LR-fuzzy number (Oussalah and De Schutter (2003)) and consequently implies further difficulty in the implementation of the fuzzy least square method for the model estimation. Another possible specification could be $\tilde{R}_i = \tilde{\alpha}_i \oplus \beta R_m \oplus \tilde{\epsilon}_i$, however it requires a real-valued random return $R_m$ of the market portfolio.

The crisp parameter $\beta_i$ is the sensitivity of the fuzzy return of the asset $i$ to the market portfolio fuzzy returns as in the standard market line. It represents the systematic risk under the above-mentioned assumptions. If the spreads of the fuzzy returns are null, then it degenerates to the slope parameter of the standard market line (CAPM).

4.2 Model estimation

The estimation problem of the linear regression model with fuzzy data has been previously treated in the literature from different point of views. Näther (2006) gives a large overview of the main approaches. In this paper, we opt for the Wünsche and Näther (2002) approach, which analyzes the problem in the particular case of convex fuzzy random variables on $\mathbb{R}^n$. Using the $\delta_2$ metric presented in Equation (4) (below), they reduce the estimation to the following optimization problem

$$\inf_{\tilde{\alpha}_i \in \mathcal{F}(\mathbb{R}), \beta \in \mathbb{R}} E(\delta_2^2(\tilde{R}_i, \tilde{\alpha}_i \oplus \beta \tilde{R}_m))$$

The solution of this least square problem is given by

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Let $\hat{R}_i$ and $\hat{R}_m$ be two square-integrable fuzzy random variables and consider the optimization problem \eqref{12}.

1. Let be $\beta \geq 0$: If $\text{Cov}(\hat{R}_i, \hat{R}_m) \geq 0$ and $\mathbb{E}^A(\hat{R}_i) \otimes_H \frac{\text{Cov}(\hat{R}_i, \hat{R}_m)}{\text{Var}(\hat{R}_m)} \mathbb{E}^A(\hat{R}_m)$ exists then

$$\beta^* = \frac{\text{Cov}(\hat{R}_i, \hat{R}_m)}{\text{Var}(\hat{R}_m)} \quad \alpha_i^* = \mathbb{E}^A(\hat{R}_i) \otimes_H \beta^* \mathbb{E}^A(\hat{R}_m)$$

2. Let be $\beta \leq 0$: If $\text{Cov}(\hat{R}_i, -\hat{R}_m) \geq 0$ and $\mathbb{E}^A(\hat{R}_i) \otimes_H -\frac{\text{Cov}(\hat{R}_i, -\hat{R}_m)}{\text{Var}(\hat{R}_m)} \mathbb{E}^A(\hat{R}_m)$ exists then

$$\beta^* = \frac{\text{Cov}(\hat{R}_i, \hat{R}_m)}{\text{Var}(\hat{R}_m)} \quad \alpha_i^* = \mathbb{E}^A(\hat{R}_i) \otimes_H \beta^* \mathbb{E}^A(\hat{R}_m)$$

are solutions of \eqref{12}, where $\otimes_H$ denotes the Hukuhara difference.

In the special case of LR-fuzzy returns, these beta estimates can be related to the beta of the classical CAPM as follows

**Proposition 4.2** Let $R_i = (R_i, \Delta_i)_{L}$ and $R_m = (R_m, \Delta_m)_{L}$ be the symmetric fuzzy returns with shape function $L$ of the asset $i$ and the market portfolio $m$, respectively. If $\beta_0$ is the systematic risk estimated from the Sharpe market line and $\beta$ the systematic risk from the fuzzy market line, then there exists a unique strictly positive real number $\lambda$ such as

$$\beta = \beta_0 + \lambda[\text{cov}(\Delta_i, \Delta_m)\text{var}(R_m) - \text{cov}(R_i, R_m)\text{var}(\Delta_m)] \quad \text{(13)}$$

Before interpreting this relationship, we first define a quantity $\beta_\Delta = \frac{\text{Cov}(\Delta_i, \Delta_m)}{\text{Var}(\Delta_m)}$. $\beta_\Delta$ determines the sensitivity of the asset intraperiod volatility (spread of fuzzy return) to the market portfolio one. Proposition \ref{4.2} then implies that

**Corollary 4.3**

$$\beta > \beta_0 \iff \beta_\Delta > \beta_0 \quad \text{(14)}$$

The equivalence \ref{14} can be interpreted as follows: If the sensitivity of the asset intraperiod volatility (spread of fuzzy return) to the market portfolio one is higher than the systematic risk estimate based on the considered period return (the standard beta), then the beta of the fuzzy market is an upward correction of the systematic risk. If there is not some linear relationship between the intraperiod volatility of the asset and the intraperiod market portfolio volatility, then our beta estimate degenerates to the classical one. This interpretation emphasizes the fact that the beta of the fuzzy market line appears as a generalization of the standard beta resulting from the incorporation of the intraperiod volatility in the return representation of an asset for a given period.

---

\[ ^{18} \text{In other words, fuzzy random variables with finite variance.} \]

\[ ^{19} \text{For two LR-fuzzy numbers } \tilde{A}_1 = (l_1, a_1, r_1) \text{ and } \tilde{A}_2 = (l_2, a_2, r_2), \text{ the Hukuhara difference } \tilde{A}_1 \otimes_H \tilde{A}_2 \text{ exists if } l_1 \geq l_2, r_1 \geq r_2 \text{ and is given by } \tilde{A}_1 \otimes_H \tilde{A}_2 = (l_1 - l_2, a_1 - a_2, r_1 - r_2). \]
4.3 Hypothesis testing on beta estimate

The goal of this subsection is to determine whether or not the systematic risk beta, introduced by the FML, vanishes. In other words, we want to assess the suitability of the assumption that the fuzzy returns of the considered asset is linearly related to market portfolio fuzzy return. This is done by testing the hypothesis of the nullity of the slope parameter of the fuzzy linear model. The subsection is largely attributable to Gil et al. (2007).

The hypotheses are formulated as follows:

\[ H_0 : \beta_i = 0 \]
\[ H_1 : \beta_i \neq 0 \] (15)

We have the following equivalent hypotheses

**Lemma 4.4** If \( \text{Cov}(R_i, R_m) \) and \( \text{Cov}(\Delta_i, \Delta_m) \) have the same sign and if \( R_m \) is not almost surely constant, then the hypotheses of (16) are equivalent to the following assertions

\[ H_0 : \text{Cov}(R_i, R_m)^2 + \text{Cov}(\Delta_i, \Delta_m)^2 = 0 \] (16)
\[ H_1 : \text{Cov}(R_i, R_m)^2 + \text{Cov}(\Delta_i, \Delta_m)^2 > 0 \]

Then, Lemma 4.4 states that, testing \( H_0 \) against \( H_1 \) reduces to testing the equivalent hypotheses about the covariance matrix of the classical random vector \( R = (R_m, R_i, \Delta_m, \Delta_i) \) on the basis of a random sample of \( n \) independent random vectors \( R_1, ..., R_n \) with \( R_j = (R_{jm}, R_{ji}, \Delta_{jm}, \Delta_{ji}) \) for \( j = 1, ..., n \).

Gil et al. (2007) developed an exact and an asymptotic method for testing the above-specified hypotheses. The exact method introduced appears to be unrealistic and in some sense limited for practical applications because of the difficulty of finding a well-supported model for the distributions of interval-valued random sets. The asymptotic method, meanwhile, is not adapted for our case because it assumes that available samples are large\(^{20}\) and takes advantage of the large sample theory. For these reasons, in what follows, we opt for the bootstrap approach to test our hypotheses. This method is presented in Appendix A.

5 Empirical studies

In this section, we empirically investigate the FML-estimated systematic risk estimate. We first analyze whether there is some linear relationship between the fuzzy return of the considered asset and the fuzzy return of the market portfolio by testing statistically the nullity of the beta estimate. Next, we assess the impacts of the interval return and the sample size on our beta estimate.

\(^{20}\)In fact, the small dataset is one cause of the fuzziness which motivates the introduction of the fuzzy market line.
estimates.

The main objective of the section is to investigate empirically the properties of fuzzy least square (FLS) beta estimates. Using our dataset, we compare the behaviors of the FLS beta, the OLS beta and the beta estimated by the Gençay et al. (2005) wavelet multiscaling method when the return interval and the sample size change.

Our data set available on Yahoo Finance France, consists of 30 stocks\textsuperscript{21} quoted in the French financial market of Paris and the CAC40 index\textsuperscript{22} is taken as the market portfolio. The sample period covers April 2007 to December 2009. Table\textsuperscript{1} lists the stocks names, their tickers used in this paper and statistics summary of monthly returns based on closing prices without including dividends.

Weakly, biweekly, triweekly and monthly fuzzy returns are successively constructed, by associating crisp return based on closing prices to intraperiodic volatility following the procedure presented in section 3. For simplicity in the analysis\textsuperscript{23} we assume that the fuzzy returns have triangular membership functions. For each return interval, the systematic risks are estimated and Figure\textsuperscript{1} compares the fuzzy least square (FLS), the ordinary least square (OLS) estimates and the wavelet estimates at different return intervals and scales. Before studying the behavior of our beta estimate when the return interval and the estimation period length change, we test the null hypothesis that it vanishes.

We use the bootstrap approaches exposed in Appendix 4.3. We apply Algorithm A.1 to compute the bootstrap p-value of the statistic $\theta^*$, using 1000 bootstrap replications. Table 2 listing of the test statistic and the 99\% fractile of the distribution of $\theta^*$, shows that the null hypothesis $H_0$ is rejected at the nominal significance level 1\% for all considered stocks, whence the relevance of the linear relationship assumed by the fuzzy one-factor market model and the suitability of our beta estimate.

The above-realized statistical hypothesis testing validates of the linear relationship assumed by the fuzzy one-factor market model and the suitability of our beta estimate. Next, in a comparative analysis, we will assess the behavior of the beta when the return interval or the length of the estimation period change. For this interval effect, the properties of the wavelet multiscaling beta, the OLS beta and the FLS beta will be compared. Since the wavelet multiscaling betas are computed using the same return interval for different period dynamics, only the OLS beta and the FLS beta will be evaluated for the impact of the sample size on the estimate.

\textsuperscript{21}These 30 stocks are those of the 40 highest market caps of CAC40 which present complete dataset over the study period.
\textsuperscript{22}The CAC40 index is a capitalization-weighted measure of the 40 highest market caps on the Paris Stock Exchange (Euronext).
\textsuperscript{23}This assumption is common in the literature. Koissi and Shapiro(2006), Andrés-Sánchez (2007) and Berry-Stolze et al. (2010) are some examples.
Table 1: This table gives descriptive statistics and the tickers of the stocks and the market portfolio index examined in the empirical studies.

Gencay et al. (2005) introduced the multiscale systematic risk to overcome the interval effect in the beta estimation. Their estimation method is based on a wavelet analysis that enables a time series to be decomposed into different time scales. For a given asset $i$, they first apply the wavelet transform to the return time series $r_{it}$ yielding the wavelet coefficient vector $w_{i}$. 

By assuming that the dependence structure of the return $r_{it}$ is independent of time, they define the time-independent wavelet variance of asset $i$ at scale $k$ to be

$$
\sigma^2_{m_j} = \text{Var}(w_{mk})
$$

Wavelet transform is a mathematical analysis tool which aiming at approximating a function of time. This approximation is made by projecting the function on a basis of wavelet functions. (See Appendix B)

Wavelet coefficient vector can be viewed as the coordinates of the analyzed function on the basis of wavelet functions.

The dependence structure refers here to the autocorrelation of the returns time series. It is independent to time when $r_{it}$ is assumed to be a stationary time series (even if the $r_{it}$ depends on $t$).
Table 2: This table emphasizes that $\theta$ statistic is strictly superior to the 99% fractile of the distribution of $\theta^*$ whence the rejection of the null hypothesis $H_0$ at the nominal significance level 1% for every stocks of the sample.

Let $r_{it}$ and $r_{jt}$ be the return for two distinct assets $i$ and $j$. Similarly, the wavelet covariance between $r_{it}$ and $r_{jt}$ for level $k$

$$
\sigma_{ij,k} = \text{Cov}(w_{ik}, w_{jk}).
$$

Finally, the wavelet beta of asset $i$ at level $k$ is estimated as follows

$$
\beta_{ik} = \frac{\sigma_{im,k}}{\sigma_{mm,k}}.
$$

where $m$ denotes de market portfolio.

For more details on the wavelet variance and covariance, see Gençay et al. (2001, Ch. 7) and references therein.

For the practical estimation of the multiscale systematic risk, we employ the daily return
time series. We use the Daubechies least asymmetric wavelet filter\(^{27}\) of length 8 to transform these time series. We estimate the wavelet beta for scales \(j = 1, \ldots, 4\) such that scale \(2^{28}\) 1 is associated with \(2 - 4\) days dynamics, scale 2 is associated with \(4 - 8\) days dynamics, scale 3 is associated with \(8 - 16\) days dynamics and scale 4 is associated with \(16 - 32\) days dynamics. In the comparative analysis, we suppose that scale 1 corresponds to weekly return interval, scale 2 corresponds to biweekly return interval, scale 3 corresponds to triweekly return interval and scale 4 corresponds to monthly return interval.

Figure 1 shows that different values of the beta can be estimated by changing the interval return over the same sample period as well as by OLS, FLS and wavelet. Figure 2 illustrates the closeness of the OLS, FLS and the wavelet beta for the different return intervals. Figure 3, which depicts the coefficient of variation\(^{29}\) of betas for different interval returns, emphasizes that the FLS estimate of beta has a smaller value of the coefficient of variation over the sample of the studied stocks, than the OLS and the wavelet beta estimates. The mean values, over all stocks and return intervals, of the coefficient of variation are 9.67% for FLS betas, 10.96% OLS betas and 13.01% in our study sample. Hence, the relative improvements reported in Table 3.

In conclusion, the use of the FLS estimation method reduces the effect of the interval return on the systematic risks estimation for the stocks used in our study.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Improvement rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>FLS over OLS</td>
<td>11.77%</td>
</tr>
<tr>
<td>FLS over Wavelet</td>
<td>25.67%</td>
</tr>
<tr>
<td>OLS over Wavelet</td>
<td>15.75%</td>
</tr>
</tbody>
</table>

Table 3: Relative improvement of FLS over OLS and wavelet method.

In order to analyze the impact of the estimation period length on the beta estimates for each interval return, we compute different betas by changing the sample size. We begin with sub-samples of the returns time series that contain the last half of observations and we increase the size of the sample by adding additional observations, one at a time.

Figure 4, depicting the beta estimates \(\beta(n)\) defined as a function of the length of the estimation period \(n\), shows that the beta estimates depend on the length of training sample for every considered interval return, for both the OLS as for FLS methods. It is tenuous to read a precise sense of variation even if we observe a kind of convergent trend toward 1.

For testing this hypothesis on the convergence of the beta estimates with sample size as

\(^{27}\)Daubechies least asymmetric wavelet filter also known as symlet, is one of the families of wavelet filters developed in Daubechies (1992, Ch. 6).

\(^{28}\)In this case, the scale \(j\) is associated with the frequency interval \([1/2^{j+1}, 1/2^j]\) and therefore, the period interval \([2^j, 2^{j+1}]\). Since we use the daily returns, it follows that the wavelet beta at scale \(j\) is associated with \(2^j - 2^{j+1}\) days dynamics.

\(^{29}\)The coefficient of variation (CV) is a normalized measure of dispersion. It is defined by \(CV = \frac{\sigma}{\mu}\), where \(\sigma\) is the standard deviation and \(\mu\) the mean value.
argument, we introduce the following linear model

$$|1 - \beta(n)| = \gamma_0 + \gamma_1 n + \epsilon_n, \ n > n_0$$

(20)

where $\beta(n)$ is the beta estimate based on an $n$-length time series, $\gamma_0$ and $\gamma_1$ are real parameters, $n_0$ is the minimal length of sample used and $\epsilon_n$ is a white noise for all feasible $n$.

Equation (20) states that there is a linear relationship between the absolute deviation from 1 of the beta estimate and the estimation sample size. Testing its validity is equivalent to testing the hypothesis of the nullity of the parameter $\gamma_0$. The hypotheses are formulated as follow

Figure 1: This plot assesses the closeness of the FLS, OLS and wavelet beta estimates for different return intervals.
Figure 2: This plot highlights the impact of the return interval on the beta estimates. We observe that different values of the beta can be estimated by changing the return interval with the same sample. The impact of the return interval on the beta estimate observed for the OLS method, also exists for the FLS method.

\[ H_0 : \gamma_1 = 0 \]
\[ H_1 : \gamma_1 \neq 0 \]  

Tables 5, 6, 7 and 8 report the results of the T-test of the linear dependence between the absolute deviation of the beta from 1 and the size of the estimation sample for different return intervals. Except the triweekly return interval, the hypothesis of the linear dependence between the absolute deviations of the standard systematic risk from 1 and the size of the estimation sample is accepted for at least 87\% of the examined stocks with weekly, biweekly and monthly returns. Moreover, according to the negative sign of the \( \gamma_1 \) estimates in the most cases, the beta estimates seem to converge weakly toward 1 when the sample size grows. For the FLS beta estimates, this behavior is observed at a relatively lower proportion (at most for 80\%) for the large return intervals (triweekly and monthly returns). However, as depicted in Figure 5, our FLS beta estimates have generally a lower coefficient of variation when the size sample change.

<table>
<thead>
<tr>
<th>Return interval</th>
<th>OLS</th>
<th>FLS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Rejection rate</td>
<td>Acceptance rate</td>
</tr>
<tr>
<td>1 week</td>
<td>87%</td>
<td>13%</td>
</tr>
<tr>
<td>2 weeks</td>
<td>87%</td>
<td>13%</td>
</tr>
<tr>
<td>3 weeks</td>
<td>37%</td>
<td>63%</td>
</tr>
<tr>
<td>4 weeks</td>
<td>90%</td>
<td>10%</td>
</tr>
</tbody>
</table>

Table 4: Rejection rate of the null hypothesis \( H_0 \) with different return intervals and estimation methods.
Figure 3: Plot of coefficients of variation of beta estimates for different return interval showing that the fuzzy least square method produces estimates with lower coefficient of variation than OLS and wavelet multiscaling approaches.
Figure 4: Plots of beta estimates ($\beta(n)$) as function of size sample $n$ with weekly (top), biweekly (second), triweekly (third) and monthly (bottom) return intervals.
6 Conclusions

This paper has focused on the effect of return intervals and on the impact of period length on beta estimates. In order to do that, the return over an interval period is first represented as a fuzzy random variable by associating the return based on closing prices to intraperiod volatility. Next, the one-factor market model is introduced as a fuzzy linear regression model and validated by a hypothesis test based on bootstrap approach. The beta from this FML is then estimated by the fuzzy least square method.

Secondly, a comparative empirical analysis based on 30 French stocks suggests that the values of FLS beta estimate and OLS beta estimates may be generally very close. They are both sensitive to the change of the return intervals and of estimation period length. However, the coefficient of variation of the FLS beta estimates is on average lower than the OLS beta estimates and the wavelet beta estimates when the returns intervals change. Moreover, the linear dependence hypothesis between the absolute deviations of the standard systematic risk from 1 and the size of the estimation sample is accepted for at least 87% of the considered stocks with weekly, biweekly and monthly returns whereas the acceptance rate is very low (between 10% and 13% for weekly, biweekly and monthly return) for the FML-based systematic risk. Triweekly returns however present contrary results with a acceptance rate at least greater than 65% for
the two methods. It also appears that the betas obtained via wavelet methods of Gencay et al. (2005) are generally quite different from the FLS and OLS based ones (which are close). To go into detail of this appearance, in future work it will be interesting to adapt the wavelet multiscaling method to fuzzy returns for computing the systematic risk. A comparative analysis with previous methods could give an explanation of observed differences between wavelet beta estimates and the two others. In conclusion, this paper has introduced a new estimation approach of the systematic risk based on the fuzzy set theory which deals better with the return interval effect and the impact of the training sample size.
A Hypothesis testing

This appendix briefly presents the bootstrap testing method of Gil et al. (2007), used for testing the nullity of the systematic risk of the fuzzy market line.

When the sample size is insufficient for straightforward statistical inference, the bootstrap approach becomes even more valuable than the one for random variables or vectors, which requires assumptions about the probabilistic distributions. Bootstrap techniques with interval data have been applied in the literature including, among others, Efron (1981) and Hall et al. (2001).

Let \((R_m, \Delta_m)\) and \((R_i, \Delta_i)\) be random vectors associated with the probability space \((\Omega, \mathcal{A}, P)\).

Let \(\Omega_l = \{\omega \in \Omega | (R_m(\omega), R_i(\omega), \Delta_m(\omega), \Delta_i(\omega)) = l\ th\ value\ of\ ((R_m, \Delta_m), (R_i, \Delta_i))\) in an arbitrary ranking considered on \((\mathcal{R}_m, \mathcal{R}_i, \Delta_m, \Delta_i)(\Omega)\) and \(P(\Omega_l; p_l) = p_l\ for\ l \in \{1, \ldots, L\}, \) where \(L\) is the cardinality of the range \((\mathcal{R}_m, \mathcal{R}_i, \Delta_m, \Delta_i)(\Omega)\). The joint probability distribution of \((R_m, \Delta_m), (R_i, \Delta_i)\) can be characterized by means of the vectorial-valued parameter \(p = (p_1, \ldots, p_{L-1})\).

Let \(((R_{m1}, \Delta_{m1}), (R_{i1}, \Delta_{i1})), \ldots, ((R_{mn}, \Delta_{mn}), (R_{in}, \Delta_{in}))\) be the \(n\) independent and identically distributed random elements determining a simple random sample from \((R_m, \Delta_m), (R_i, \Delta_i)\). Let \(f_{nl}\) denote the statistic representing the sample relative frequency of the \(lth\ 'value'\ of \(((R_m, \Delta_m), (R_i, \Delta_i))\) in the considered ranking and let \(\mathbf{f}_n = (f_{n1}, \ldots, f_{n(l-1)})\) be the estimator of \(p\).

Beran and Srivastava (1985) have designed a valuable bootstrap method for dealing with random vectors. Their method can then be applied to test \(H_0, \) more precisely the nullity of the four elements \(\sigma_{12}, \sigma_{34}, \sigma_{21}\) and \(\sigma_{43}\) of the covariance matrix \(\Sigma = (\sigma_{ij})_{ij} \) (of order \(4 \times 4\)) of the random vector \((R_m, R_i, \Delta_m, \Delta_i)\) defined by

\[
\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}
\]  

with

\[
\Sigma_{11} = \begin{pmatrix} \text{Var}(R_m; \mathbf{p}) & \text{Cov}(R_m, R_i; \mathbf{p}) \\ \text{Cov}(R_m, R_i; \mathbf{p}) & \text{Var}(R_i; \mathbf{p}) \end{pmatrix}
\]  

\[
\Sigma_{12} = \begin{pmatrix} \text{Cov}(R_m, \Delta_m; \mathbf{p}) & \text{Cov}(R_m, \Delta_i; \mathbf{p}) \\ \text{Cov}(\Delta_m, R_i; \mathbf{p}) & \text{Cov}(\Delta_i, R_i; \mathbf{p}) \end{pmatrix}
\]  

\[
\Sigma_{21} = \begin{pmatrix} \text{Cov}(R_m, \Delta_m; \mathbf{p}) & \text{Cov}(R_i, \Delta_m; \mathbf{p}) \\ \text{Cov}(\Delta_m, \Delta_i; \mathbf{p}) & \text{Cov}(\Delta_i, \Delta_i; \mathbf{p}) \end{pmatrix}
\]  

\[
\Sigma_{22} = \begin{pmatrix} \text{Var}(\Delta_m; \mathbf{p}) & \text{Cov}(\Delta_m, \Delta_i; \mathbf{p}) \\ \text{Cov}(\Delta_m, \Delta_i; \mathbf{p}) & \text{Var}(\Delta_i; \mathbf{p}) \end{pmatrix}
\]
\[
\Sigma_{22} = \begin{pmatrix}
\text{Var}(\Delta_m; p) & \text{Cov}(\Delta_m, \Delta_i; p) \\
\text{Cov}(\Delta_m, \Delta_i; p) & \text{Var}(\Delta_i; p)
\end{pmatrix}
\] (26)

Gil et al. (2007) apply the method by Beran and Srivastava (1985) as presented in Algorithm A.1 for computing the fractile of the bootstrap statistic \( \theta^* \).

Algorithm A.1

**Step 1**: Compute the variance-covariance matrix of \( Z_j = (R_{mj}, R_{ij}, \Delta_{mj}, \Delta_{ij}) \) for \( j = 1, \ldots, n \)

\[
\hat{S} = \begin{pmatrix}
\hat{S}_{11} & \hat{S}_{12} \\
\hat{S}_{21} & \hat{S}_{22}
\end{pmatrix}
\] (27)

where

\[
\hat{S}_{11} = \begin{pmatrix}
\text{Var}(R_m; f_n) & \text{Cov}(R_m, R_i; f_n) \\
\text{Cov}(R_m, R_i; f_n) & \text{Var}(R_i; f_n)
\end{pmatrix}
\] (28)

\[
\hat{S}_{12} = \begin{pmatrix}
\text{Cov}(R_m, \Delta_m; f_n) & \text{Cov}(R_m, \Delta_i; f_n) \\
\text{Cov}(\Delta_m, R_i; f_n) & \text{Cov}(R_i, \Delta_i; f_n)
\end{pmatrix}
\] (29)

\[
\hat{S}_{21} = \begin{pmatrix}
\text{Cov}(R_m, \Delta_m; f_n) & \text{Cov}(R_i, \Delta_m; f_n) \\
\text{Cov}(R_m, \Delta_i; f_n) & \text{Cov}(R_i, \Delta_i; f_n)
\end{pmatrix}
\] (30)

\[
\hat{S}_{22} = \begin{pmatrix}
\text{Var}(\Delta_m; f_n) & \text{Cov}(\Delta_m, \Delta_i; f_n) \\
\text{Cov}(\Delta_m, \Delta_i; f_n) & \text{Var}(\Delta_i; f_n)
\end{pmatrix}
\] (31)

and the value of statistic \( \Theta = n h(\hat{S}) \), where

\[
h(\hat{S}) = \text{Cov}(R_m, R_i; f_n)^2 + \text{Cov}(\Delta_m, \Delta_i; f_n)^2
\] (32)

**Step 2**: Compute

\[
\pi(\hat{S}) = \begin{pmatrix}
\pi_{11}(\hat{S}_{11}) & \hat{S}_{12} \\
\hat{S}_{21} & \pi_{11}(\hat{S}_{22})
\end{pmatrix}
\] (33)

where

\[
\pi_{11}(\hat{S}_{11}) = \begin{pmatrix}
\text{Var}(R_m; f_n) & 0 \\
0 & \text{Var}(R_i; f_n)
\end{pmatrix}
\] (34)

\[
\pi_{22}(\hat{S}_{22}) = \begin{pmatrix}
\text{Var}(\Delta_m; f_n) & 0 \\
0 & \text{Var}(\Delta_i; f_n)
\end{pmatrix}
\] (35)

**Step 3**: Compute the bootstrap population

\[
V_j = (Z_j \hat{S}^{-1/2} \pi(\hat{S})^{1/2}) = (V^1_j, V^2_j, V^3_j, V^4_j), \ \forall \ j = 1, \ldots, n
\] (36)

**Step 4**: Obtain a sample of independent and identically distributed random vectors \((V^*_1, \ldots, V^*_n)\) from the bootstrap population, with \( V^*_j = (V^{1*}_j, V^{2*}_j, V^{3*}_j, V^{4*}_j), \ \forall \ j = 1, \ldots, n \).
**Step 5**: Compute the value of bootstrap statistic

\[ \theta^* = n \left[ \text{Cov} \left( V_1^*, V_2^*; f_n^* \right)^2 + \text{cov} \left( V_3^*, V_4^*; f_n^* \right)^2 \right] \]  

(37)

being \( f_n^* = (1/n, ..., 1/n) \)

**Step 6** Steps 4 and 5 should be repeated a large number \( B \) of times to get a set of \( B \) estimators, denoted by \( \{ \theta^*_1, ..., \theta^*_B \} \).

**Step 7** Compute the bootstrap p-value as proportion of values in \( \{ \theta^*_1, ..., \theta^*_B \} \) being greater than \( \theta \).

Finally, the decision rule is given by

**Theorem A.2** (Gil et al.(2007)) To test at nominal significance level \( \gamma \) the null hypothesis \( H_0 : \beta = 0 \) against alternative \( H_1 : \beta \neq 0 \), \( H_0 \) should be rejected if

\[ \theta = nh(\hat{S}) > c_\gamma \]  

(38)

where \( c_\gamma \) is the \( 100(1 - \gamma) \) fractile of the distribution of \( \theta^* \) and \( n \) the number of observations.
B Wavelet analysis

This appendix briefly presents wavelet analysis. The concepts of orthogonal wavelet bases and wavelet transform are reviewed. This review is adapted from Subsection 2.1. of Vanucci and Corradi (1999).

Wavelets are families of functions which can accurately approximate another function. This approximation is made by projecting this function on a basis of wavelet functions.

In $L^2(\mathbb{R})$ a wavelet basis is obtained by translations and dilatations of a scaling function $\phi$, constructed as solution of a solution of a dilation equation

$$\phi(t) = \sqrt{2} \sum_{l \in \mathbb{Z}} h_l \phi(2t - l)$$

where a mother wavelet $\psi$, defined from $\phi$ as $\psi(t) = \sqrt{2} \sum_{l \in \mathbb{Z}} g_l \phi(2t - l)$ with filter coefficients $g_l$ defined by $g_l = (-1)^l h_{1-l}$.

The wavelet collection is obtained by translations and dilations as

$$\phi_{j,k}(t) = 2^{j/2} \phi(2^j t - k)$$
$$\psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k)$$

The family of wavelets $\{\psi_{j,k}, \; j, k \in \mathbb{Z}\}$ forms an orthonormal basis in $L^2(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}} \psi_{j',k'}(t) \psi_{j,k}(t) dt = \begin{cases} 1 & \text{if } j' = j \text{ and } k' = k, \\ 0 & \text{elsewhere.} \end{cases}$$

Any $L^2(\mathbb{R})$ function $f$ can be written by a wavelet series as $f(t) = \sum_{j,k \in \mathbb{Z}} d_{j,k} \psi_{j,k}(t)$ with wavelet coefficients

$$d_{j,k} = \int_{\mathbb{R}} \psi_{j,k}(t) f(t) dt$$

The wavelet coefficients which are the coordinates of the function $f$ in the wavelets basis, describe features of $f$ at the spatial location $2^{-j}k$ and scale $j$.

In a similar way, the scaling coefficients can be defined using the scaling functions as follows:

$$c_{j,k} = \int_{\mathbb{R}} \phi_{j,k}(t) f(t) dt$$

$L^2(\mathbb{R})$ is the space of functions $f$ such that

$$\int_{\mathbb{R}} |f(t)|^2 dt < \infty$$

26
Recursive relationships between wavelet coefficients at scale $j$ and scaling coefficients at the finer scale $j + 1$ can be obtained as

$$c_{j,k} = \sum_{m \in \mathbb{Z}} h_{m-2k} c_{j+1,m},$$  \hspace{1cm} (45)$$

$$d_{j,k} = \sum_{m \in \mathbb{Z}} g_{m-2k} c_{j+1,m}.$$  \hspace{1cm} (46)

These equations are used to derive a fast algorithm, known as the discrete wavelet transform (DWT), for decomposing a function into a set of wavelet coefficients.

Readers interested in a detailed presentation of wavelet analysis, may see Daubechies (1992).
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Table 5: Results of the testing the nullity hypothesis of $\gamma_1$ based on monthly returns with beta estimated by OLS and FLS.
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Table 6: Results of the testing the nullity hypothesis of $\gamma_1$ based on triweekly returns with beta estimated by OLS and FLS.
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Table 7: Results of the testing the nullity hypothesis of $\gamma_1$ based on biweekly returns with beta estimated by OLS and FLS.
Table 8: Results of the testing the nullity hypothesis of $\gamma_1$ based on weekly returns with beta estimated by OLS and FLS.
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Stéphane MUSSARD: mussard@lameta.univ-montp1.fr