Exhaustible Resources and Fully-Endogenous Growth with Non-Scale Effects✩

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Abstract

R&D-based endogenous growth theory predicts long-run growth dependent on the size of the economy, although the empirical studies criticize this scale effect. The models are also theoretically criticized because of the lack of robustness linked to the knife-edge assumption of constant returns to scale to producible inputs. Semi-endogenous growth literature shows that permanent growth is still feasible with diminishing returns if it is based on an exogenous growing factor, usually population growth. We prove that sustained growth under increasing returns is feasible when final output depends on non-renewable resources, whose use necessarily shrinks in the long run. The instability problem associated with increasing returns is overcome when the final output and the R&D sectors are separately considered, and the latter depends, even indirectly, on the natural resource. Economic growth is fully endogenous because no exogenous source of growth is needed. Furthermore no knife-edge assumption is required and growth shows non-scale effects.

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1. Introduction

The first generation of R&D-based endogenous growth models predicted that the long-run growth rate of per capita income was affected by the size of the population. This is what has been called “strong scale effects” in the economic growth literature, which has little empirical support (see Backus et al., 1994; Jones, 1995a). These models have also been theoretically criticized because of the so-called knife-edge condition. That is, the assumption of exactly constant returns to scale (CRS) in the accumulated factors of production in the R&D sector. As Solow (1994) points out, the knife-edge condition implies a lack of robustness. Slightly increasing returns to scale (IRS) would lead to explosive growth, while diminishing returns would lead to stagnation. The knife-edge condition is alleviated and the strong form of scale effects are successfully removed by the semi-endogenous growth models, allowing for diminishing returns to scale (DRS). However, these models predict the weak form of scale effects, since a permanent economic growth must be supported by an exogenously growing factor, usually population growth. This is why these models are known as semi-endogenous.

This paper studies an R&D-based endogenous growth model with the use of a non-renewable natural resource as a productive essential input. The knife-edge assumption is avoided and any form of scale effects (weak or strong) is absent. The most relevant feature is that the consumption goods sector is separated from the R&D or research sector which is driven by monopolistic entrepreneurship. Setting the economy in a decentralized framework with imperfect markets, not only guarantees the appropriate reward for the research efforts, but also the stability of a sustained long-run growth in absence of population growth, with far reaching policy implications.

A first group of papers that analyze R&D-based endogenous growth models and include non-renewable natural resources as a necessary factor in the production of con-

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1As Jones (2005) defines, “strong scale effects” arise when the growth rate of the economy is an increasing function of scale (which typically means population). In models that exhibit “weak scale effects”, the level of per capita income in the long run is affected, in some sense, by the size of the economy.
umption goods are Scholz and Ziemes (1999), Schou (2000, 2002), and Grimaud and Rougé (2003). These authors consider a separate research sector that does not depend on the natural resource. The need to harvest successively smaller amounts of the resource allows for IRS to accumulated factors of production in the final output sector. Nevertheless, there still is a knife-edge condition in the R&D sector, represented by degree one spillovers for technical knowledge. As a consequence, the strong form of scale effects are present.\footnote{Although these authors normalize total labor to one, Groth (2007) highlights the scale effects for this type of models.}

Scale-invariant endogenous growth models are gaining interest in the economic growth literature. For instance, Dalgaard and Kreiner (2001) and Strulik (2005), employ monopolistic competition models with ever increasing average human capital levels, while Grossman (2009) proposes a model with price-taking entrepreneurial firms which invest in productivity-enhancing R&D.\footnote{Conditions for non-scaled balanced growth are analyzed in Eicher and Turnovsky (1999).}

From a different point of view, the controversial scale effects on growth may disappear when the natural resource is not only an input for final output, but also directly or indirectly, for knowledge creation, what Groth (2007) calls a \textit{growth-essential} resource. As for the knife-edge criticism, the unavoidable decay in the stock of the exhaustible resource leaves room for IRS to producible inputs, contrary to the DRS assumption in semi-endogenous growth models. Nevertheless, as Groth and Schou (2002) show, for a one sector economy, IRS to producible inputs leads to instability of the steady-state equilibrium unless the population is assumed to grow at a positive constant rate. Thus, with no growth in population, the knife-edge condition translates into an instability problem.

Groth and Schou (2002, 2007) leave an open question: to what extent could the instability problem be overcome if a separate R&D sector were considered? Our paper gives a positive answer to this question. We prove that, even without exogenous population growth, a non-scaled stable equilibrium with positive growth is attained. Sustained economic growth is not restricted to the knife-edge assumption of CRS and furthermore,
it is fully endogenous because it does not rely on an exogenous source of growth as, population growth is in the semi-endogenous growth literature.

The existence of a steady-state equilibrium requires that producible inputs exhibit IRS in the research sector when they show CRS in the consumption good sector. Alternatively, if they exhibited IRS in the consumption goods sector, sustained positive growth would still be feasible for IRS, CRS or even slightly DRS in the research sector. In both cases the elasticity of the intertemporal substitution of consumption cannot be excessively small. Under these conditions the steady-state equilibrium, where the economy grows indefinitely is unique and does not exhibit any scale effects. As for the question of stability, the equilibrium is saddle-path stable taking into account two assumptions. First, imperfect competition in the decentralized economy and second, two separated sectors for output and knowledge. Under these two assumptions the equilibrium is saddle-path stable for a non-empty open set of values for the elasticity of capital in the R&D sector.

The paper is organized as follows. Section 2 describes a quite general R&D-based growth model with a growth-essential exhaustible natural resource. In Section 3 the growth rates for the main variables are characterized along the balanced path. Conditions for the existence and uniqueness of this equilibrium are provided. In Section 4 the saddle-path stability of the balanced path equilibrium is proved. The transitional adjustment towards this equilibrium after an exogenous shock on the stock of the resource is also analyzed. Finally, Section 5 concludes.

2. An R&D-based growth model with a growth-essential exhaustible natural resource

R&D-based endogenous growth models rely on knowledge creation and technological change to keep the productivity of neoclassical inputs (capital and labor) growing. When a non-renewable natural resource is a productive input, technical progress is also needed to overcome the continuous decline in the use of this resource. Specifically, in the economy considered here, knowledge creation and final output production occur in two separate
sectors. The following equations summarize the main features of an R&D-based model of economic growth with an exhaustible natural resource sector, in a way as general as possible:

\[ Y = AN^{\alpha}K^{\beta}Y^{1-\alpha-\beta}R^{\beta}, \quad A > 0, \quad \alpha, \beta \in (0, 1), \quad \alpha + \beta < 1, \quad \varepsilon \geq 1 - \alpha, \quad (1) \]

\[ \dot{N} = BN^{\varphi}K^{\eta}N^{1-\eta}, \quad N(0) = N_0, \quad B > 0, \quad \eta, \varphi \in (0, 1), \quad (2) \]

\[ \dot{K} = Y - C, \quad K(0) = K_0, \quad (3) \]

\[ \dot{S} = -R, \quad S(0) = S_0, \quad (4) \]

where \( Y \) is final output (a composite good used either for consumption or for investment in physical capital) and \( N \) is technology or knowledge. Capital and labor are either employed to produce output (\( K_Y \) and \( L_Y \)) or to search for new knowledge (\( K_N \) and \( L_N \)), where \( K = K_Y + K_N \) and \( L = L_Y + L_N \). \( C \) is the aggregated consumption, \( S \) the stock of the non-renewable resource and \( R \) the total harvesting of extracting companies.

We shall assume that the total labor force, \( L \), is constant. This is done to emphasize that this model does not need any exogenous growing factor to generate a positive growth rate.

Behind this four-equation R&D-based growth model, we are considering a quite general model of an expanding variety of intermediate goods. Since this type of model is well established in the literature, the micro-foundations are given in Appendix A. Here, we briefly comment that equations (1)-(4) emerge from a four-sector economy. The final-output sector produces the consumption/capital good using labor, an exhaustible natural resource and a set of durable intermediate goods. Extracting companies harvest a non-renewable resource in the resource sector. In the intermediate-goods sector, monopoly firms transform capital into durable goods using the designs created in the R&D sector. Discoveries are non-rival goods which are non-excludable within the innovative sector and, hence, researchers can take advantage of the existing stock of knowledge. Conversely, they are excludable in the intermediate-goods sector where monopolistic competitors buy these designs. Innovation does not only depend on the number of researchers

\[ ^4 \text{All variables depend on the time argument, omitted when no confusion can arise.} \]
and the existing stock of technology, but also on the capital stock (or foregone output) employed in this sector.

The standard model of an expanding variety of intermediate goods à la Romer (1990) (which can be found in most Economic Growth textbooks, see Barro and Sala-i-Martin, 2004, or Acemoglu, 2009) considers a multi-sector economy. The aggregate output and knowledge production for this model takes the form (1) and (2) with $\eta = 0$, $\varphi = 1$, $\varepsilon = 1 - \alpha$ and $\beta = 0$. Keeping all other parameters unchanged, a positive $\beta$ would add the non-renewable resource as a third input in the final output sector, as is presented in Scholz and Ziemes (1999).

Three characteristics in equations (1) and (2) must be highlighted because they give the model a more general scope than the previous models à la Romer, either with or without a resource sector.

First, we assume $0 < \varphi < 1$. Like most of the current literature, we assume a positive knowledge externality, hence the rate of innovation increases with the stock of knowledge, $\varphi > 0$. Standard R&D growth models have assumed $\varphi = 1$, which has been the key requirement for unbounded growth. This is an arbitrary “knife-edge” assumption which predicts strong scale effects. We follow Jones (1995b) assuming $\varphi < 1$.

Secondly, assuming $\eta \in (0, 1)$ means that the creation of new knowledge is no longer proportional to labor,\(^5\) and makes innovation also dependent on machinery used by scientists (computers, laboratories, books, accommodations, etc), i.e. capital goods. Because physical capital is produced in the manufacturing sector where the resource is an input, innovation is indirectly affected by the resource. The resource is growth-essential, as stated by Groth (2007). This property will allow for economic growth with non-scale effects.\(^6\)

Thirdly, we assume $\varepsilon \geq 1 - \alpha$. The equality $\varepsilon = 1 - \alpha$ represents the standard model

\(^5\)The idea of diminishing returns on labor in the R&D sector, $\eta < 1$, was already suggested by Romer (1990). Increasing the number of researchers would not increase new discoveries in the same proportion. The duplication and overlap of research would imply a less than proportional increment in the number of innovations.

\(^6\)The results will be essentially the same if the resource is a direct input in the growth-engine sector.
with an expanding variety of intermediate goods à la Romer. This model assumes no knowledge spillovers in the production of intermediate goods. One unit of capital can be converted into one unit of intermediate goods, implying a technology which is labor and resource augmenting in the final output sector.\(^7\) Inequality \(\varepsilon > 1 - \alpha\) encompasses the case where positive knowledge spillovers are also present in the intermediate-goods sector. Both the standard model and the model with positive knowledge spillovers in the intermediate-goods sector are presented in Appendix A in a unified framework.\(^8\)

It is well known that the optimal behaviour of all different agents in this economy with imperfect competition in the intermediate-goods sector can be summarized by three rules:

**The Ramsey rule** says that the relation between the rate of return, \(r\), and the rate of time preference, \(\rho\), determines whether households choose a growing pattern of consumption over time, a constant one or a falling one. A lower willingness to substitute present for future consumption implies a smaller responsiveness of the growth rate of consumption to the gap between \(r\) and \(\rho\). That is,

\[
\gamma_C = \frac{1}{\sigma} (r - \rho),
\]

with \(\sigma\) being the inverse of the elasticity of the intertemporal substitution of consumption.

Here, and henceforth, \(\gamma_x\) denotes the growth rate of variable \(x\).

This expression stems from farsighted consumers who can invest in assets and have a utility function with a constant elasticity of intertemporal substitution.

**The Hotelling rule** states that the marginal gains from current harvesting and from leaving the resource in the ground should be equal at the equilibrium. Thus, when harvesting is carried out at no cost, the price of the resource, \(p_R\), times the rate of return

\(^7\)Note that for \(\varepsilon = 1 - \alpha\), the production of final output could be written as:

\[
Y = AK^\alpha Y (L)^{1-\alpha} (N)^{\beta} (R)^{\beta}.
\]

\(^8\)Alternatively, it could be argued that new discoveries make the production of existing intermediate goods more arduous, which would imply \(\varepsilon < 1 - \alpha\).
on capital, $r$, should equate the increment of this price per unit of time. That is,

$$\gamma_{pR} = r.$$  \hspace{1cm} (6)

According to this intertemporal efficiency condition, the discounted value of the resource should be constant through time.

The rate of return on capital is lower than the social rate of return. As is known, this is due to the existence of imperfect competition. Monopolistic producers of intermediate goods know the demand function and charge a price over the marginal cost or competitive price. This, together with the homogeneity of the final output production function in (1), leads to

$$r = \alpha \frac{\partial Y}{\partial K} = \alpha^2 \frac{Y}{K_Y}. \hspace{1cm} (7)$$

The marginal cost of the intermediate-good producer, $r$, is shorter than the marginal productivity of the capital stock in the final output sector. Thus, the capital is underpaid, compared to the competitive case, in order to compensate the investment in the R&D sector.

The detailed derivations of these three rules can be found in Appendix A.

3. Existence and uniqueness of a balanced growth equilibrium

In the perfect-foresight equilibrium, all agents take as given the time paths of variables out of their control. Equilibrium is then characterized by equating supply and demand for all relevant quantities.

At this point we restrict our attention to the perfect-foresight balanced growth equilibrium, in which the growth rates of all variables are constant. In particular we center our attention on equilibria with a positive consumption growth, including the consideration that the natural resource is bounded. Since production depends on the exhaustible resource, an endless growth in production and consumption, jointly with a declining use of the resource, requires a continuous rise of knowledge. That is, the reduction in the use of the non-renewable resource needs to be overcompensated by the technological progress.
The next proposition characterizes the growth rates of all different variables along the balanced growth equilibrium. Let $u = L_Y/L$ and $v = K_Y/K$ be the shares of labor and capital devoted to the production of final output with $u, v \in (0, 1)$.

**Proposition 1.** Along the balanced path, the output, the consumption and the capital stocks used in the final output and the innovative sectors grow at the same constant rate. Furthermore, the shares of labor and capital in the innovative and the final output sectors and the rate of return remain constant.

\[
\begin{align*}
\gamma^* & = \gamma^*_k = \gamma^*_c = \gamma^*_y = \gamma^*_r = 0, \\
\gamma^*_u & = \gamma^*_{1-u} = \gamma^*_v = \gamma^*_{1-v} = \gamma^*_r = 0.
\end{align*}
\]

**Proof.** The share of labor devoted either to manufacturing, $u$, or to innovation, $1 - u$, is bounded, and hence, it remains constant along a balanced path, that is, $\gamma_u^* = \gamma_{1-u}^* = 0$. The same argument applies to the share of capital employed in the consumption-goods sector, $v$, and the R&D sector, $1 - v$. Furthermore, from (5) it follows that the rate of return on capital must also be constant, $\gamma^*_r = 0$, in order to have a constant growth rate for consumption. Finally, taking into account equations (3) and (7), along the balanced path, the following equalities must be fulfilled:

\[
\gamma^* = \gamma^*_k = \gamma^*_c = \gamma^*_y.
\]

To characterize the growth rate of the economy along this balanced path we derive two equations relating the growth rate of the economy and the growth rate of harvesting. The first equation stems from the assumed functional forms of the production functions in the final output and the R&D sectors. The second equation comes from the optimal microeconomic decisions of consumers (Ramsey rule), final output producers, and harvesting firms (Hotelling rule).

From the production function (1) we get:

\[
\gamma_Y = \varepsilon \gamma_N + \alpha \gamma_{K_Y} + (1 - \alpha - \beta)(\gamma_u + \gamma_L) + \beta \gamma_R. 
\]

Because $L$ is constant, $\gamma_u = 0$ and $\gamma_{K_Y} = \gamma_Y = \gamma$ along the balanced path, the above equation allows us to write:

\[
\gamma = \frac{1}{1 - \alpha} \left( \varepsilon \gamma_N + \beta \gamma_R \right).
\]
Given the finiteness of the resource, an economy whose use of resource is diminishing has to rely on technology to go beyond “the limits of growth” Meadows et al. (1972). As equation (9) shows, the extent to which technology must grow to overcome the reduction in the use of the resource depends on $\beta/\varepsilon$, the ratio of output elasticities of resource and knowledge in (1).

Technology grows at a constant rate along the balanced path and, therefore, from equation (2), the growth rate of final output and knowledge must be proportional:

$$\gamma = \frac{1 - \varphi}{\eta} \gamma_N.$$  \hspace{1cm} (10)

Taking into account (9) and (10), the first relationship between the growth rate of the economy and the (negative) growth rate of harvesting reads:

$$\gamma = \frac{(1 - \varphi)\beta}{(1 - \alpha)(1 - \varphi) - \varepsilon \eta} \gamma_R.$$  \hspace{1cm} (11)

A growing technology along the balanced path ensures a positive growth rate of the economy only if we assume $\varphi < 1$ (see equation (10)). Moreover, a balanced path equilibrium requires a decreasing use of the natural resource which from (11) is only feasible under the assumption $\varphi > 1 - \varepsilon \eta/(1 - \alpha)$. Therefore, a positive growth in manufacturing together with a decrease in the use of the resource is only feasible under the following condition, assumed henceforth.

**Condition 1:** $1 - \frac{\varepsilon \eta}{1 - \alpha} < \varphi < 1$.

Condition 1 states that returns to scale to producible inputs must be “globally” increasing, globally meaning that the research and the final output sectors are considered together. In the standard model of an expanding variety of intermediate goods, $\varepsilon = 1 - \alpha$, reproducible inputs show CRS in the final output sector and hence, according to condition 1, IRS are required in the innovative sector. Conversely, when knowledge spillovers appear in the intermediate-goods sector, $\varepsilon > 1 - \alpha$, reproducible inputs exhibit IRS in the final output sector. In this case, Condition 1 is compatible with a research sector with IRS, CRS, or even DRS, as long as the decreasing returns in this sector are not too large with respect to the increasing returns in the final output sector.
Because of perfect competition in the final output sector, the price of the resource and its marginal productivity must be equal. Therefore, the gap between the growth rates of output and harvesting equates the growth rate of the price of the resource:

\[ \gamma - \gamma_R = \gamma_{pR}, \]  

(12)

Taking into account the Ramsey and Hotelling rules, (5) and (6), in (12) the second equation relating the growth rate of the economy and the growth rate of harvesting can be rewritten as:

\[ \gamma = -\frac{\gamma_R}{\sigma - 1} - \frac{\rho}{\sigma - 1}. \]  

(13)

Both equations (11) and (13) are depicted in Figure 1. Because of condition 1, equation (11) is depicted as a downward-sloped solid line. Equation (13), depicted as a dashed line, shows a negative slope when the elasticity of intertemporal substitution of consumption is lower than one \((\sigma > 1)\); and a positive slope in the opposite case \((\sigma < 1)\). The point where both functions cross meets the assumptions made for production functions and the optimal behaviour of economic agents.

Condition 1 is not sufficient to guarantee that the equilibrium is always placed in the second quadrant, with growing production and decreasing harvesting. An additional restriction must be imposed on parameters:

9This expression is valid for \(\sigma \neq 1\). For a logarithmic utility function \((\sigma = 1)\) following the same reasoning, the alternative expression \(\gamma_R = -\rho\) is obtained.
**Condition 2:** \( \sigma < 1 - \frac{(1 - \alpha)(1 - \varphi) - \varepsilon \eta}{\beta(1 - \varphi)}. \)

Condition 2 requires that consumers are not excessively inelastic. That is they are not too reluctant to postpone consumption.\(^{10}\) When sustained growth does not rely on any exogenous source of growth as population growth, savings from highly inelastic consumers would not be sufficient to promote the necessary investment in technological improvements. Thus, sustained growth with a limited resource would not be possible.

Following the previous analysis, the next proposition characterizes the growth rate of the economy along the balanced path.

**Proposition 2.** *Under conditions 1 and 2, the growth rate of the economy along the balanced path equilibrium is given by:*

\[
\gamma^* = \frac{\rho}{1 - \sigma - \frac{(1 - \alpha)(1 - \varphi) - \varepsilon \eta}{\beta(1 - \varphi)}} > 0. \tag{14}
\]

Given limited knowledge spillovers in the research sector, as is stated in condition 1, and not excessively intertemporally inelastic consumers such that condition 2 is satisfied, then Proposition 2 characterizes a positive growth rate of the economy which does not require any exogenous source of growth.\(^{11}\)

Once the growth rate of the economy along the balanced path has been characterized, we move on to study the existence and uniqueness of such an equilibrium where all variables behave as in Proposition 1, and take values within their feasible regions.

From Proposition 1, \( u \) and \( v \) must be constant along the balanced path. Furthermore, the connection between the final output and the R&D sectors implies a relationship between these two variables. The marginal returns to labor must coincide in the R&D and the final output sectors, given in (25) and (33). As for the capital, its rate of return, \( r \), coincides with the marginal return to capital in the R&D sector, as shown in (34), while it is related to the marginal return in the final output sector by equation (7). These two equalities determine a one-to-one positive relationship between \( u \) and \( v \), given by

\[
\frac{\alpha^2}{1 - \alpha - \beta} \frac{1 - v}{v} = \frac{\eta}{1 - \eta} \frac{1 - u}{u}. \tag{15}
\]

\(^{10}\)Under condition 1, the upper bound for \( \sigma \) in condition 2, is greater than one, and therefore, condition 2 is not excessively restrictive.

\(^{11}\)If population growth was added, positive growth in consumption per capita would also be feasible.
Therefore, in the following reasoning, we shall refer to $v$ as the function of variable $u$ that satisfies equation (15).

Taking into account Proposition 1, a balanced growth path corresponds to a steady state for variables $c = C/K, u, \chi = \gamma N, y = Y/K$, and $R/S$. The evolution of these variables is obtained in Appendix B and is described by the following system of five differential equations:

\[
\begin{align*}
\gamma_c &= \left( \frac{\alpha^2}{\sigma v} - 1 \right) y + c - \frac{\rho}{\sigma}, \\
\gamma_u &= \Phi(u) \left\{ \left( 1 - \frac{\alpha}{v} - \frac{1 - \beta}{\alpha} \eta \right) y + \left( \frac{1 - \beta}{\alpha} \eta - 1 \right) c + \Gamma(u) \chi \right\}, \\
\gamma_\chi &= \Psi(u) \gamma_u + \eta y - \eta c + (\varphi - 1) \chi, \\
\gamma_y &= \Omega(u) \gamma_u + \left( \alpha - 1 - \frac{\alpha \beta}{1 - \beta} \right) \frac{\alpha}{v} y + \frac{1 - \alpha - \beta}{1 - \beta} c + \frac{\varepsilon}{1 - \beta} \chi, \\
\gamma_R &= \Omega(u) \gamma_u + \alpha \left( 1 - \frac{\alpha}{v} \right) y + \frac{\varepsilon}{1 - \beta} \chi + \frac{R}{S} - \frac{\alpha}{1 - \beta} c,
\end{align*}
\]

where

\[
\begin{align*}
\Phi(u) &= \frac{\alpha(1 - u)}{[(1 - \beta)\eta - \alpha](u - v)}, \\
\Gamma(u) &= \frac{\varepsilon - (1 - \beta)}{\alpha} - \frac{(\varphi - 1)(1 - \beta)}{\alpha} + \frac{(1 - \eta)(1 - \beta)(1 - \alpha)}{1 - \alpha - \beta} \frac{u}{1 - u}, \\
\Psi(u) &= \frac{\eta v + (1 - \eta)u}{1 - u}, \\
\Omega(u) &= \frac{\alpha(1 - v) + (1 - \alpha - \beta)(1 - u)}{(1 - \beta)(1 - u)}.
\end{align*}
\]

**Proposition 3.** Under conditions 1 and 2, there exists a unique steady-state equilibrium for the economy with $u^* \in (0, 1)$ and $c^*, y^*, \chi^*, (R/S)^* > 0$.\footnote{Note that the unique steady-state equilibrium satisfies the transversality conditions for the dynamic optimization problems of consumers and extracting companies. Transversality conditions, together with the concavity of functions, assure that the steady-state equilibrium is an optimal solution for consumers and extracting companies. The rest of the agents involved in this model solve static optimization problems.}

**Proof.** See Appendix B. \(\blacksquare\)

**Taxes and Subsidies.** As pointed out by Romer (1995) taxes and subsidies can only affect long-run growth as long as they affect a linear differential equation connected to the final
output sector. In our model, the differential equations that govern the dynamics of the stocks of capital and technology are non linear. Conversely, the dynamics of the resource stock along the balanced path can be regarded as a linear differential equation if instead of harvesting, we focus on the depletion rate, $R/S$, which is constant in the long-run. That is:

$$\dot{S} = -\left(\frac{R}{S}\right)S.$$ 

In consequence conventional policies, like interest income taxes or research subsidies, have no effect on long-run growth. In contrast, those policies that affect the extraction rate will have an effect on the long-run growth rate. Groth and Schou (2007) proved this point for two policies: a profits tax on resource extracting companies and a time-dependent tax on the price of the resource. Note finally, that our model is fully endogenous and yet still, long-run growth is policy invariant with respect to the conventional stimulus on savings or innovation. This is in contrast with the policy-dependency of fully-endogenous R&D-based growth models with linearly-evolving capital or technology.

The Social Planner’s Problem. As is usual, Pareto optimality can be assessed by comparing the previous decentralized outcomes with the results from the problem solved by a hypothetical social planner. The social planner seeks to maximize the stream of discounted utility of households, given by

$$\max_{u,v,C,R} \int_0^\infty \frac{(C(t))^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t} dt$$

subject to constraints (1)-(4). The results for the social planner’s problem are presented in Appendix C.\footnote{This appendix is technical and its objective is to ease the understanding of the social planner’s problem. However, it could be removed from the final version of the paper if the editor or the reviewers so advise.} This appendix reveals that the long-run growth rate in the decentralized economy is the same as in the social optimum, despite the presence of externalities in the R&D and in the intermediate-goods sectors and imperfect competition. This result replicates what usually happens in R&D-based growth models (see Jones, 1995b, and
Steger, 2005, among many others). However, differences appear in the steady-state values of control and state variables.

Non-rivalry of knowledge in R&D-based models of economic growth is linked to the appearance of IRS, which conflicts with perfect competition, where factors are paid according to their marginal products. As is well known, the compensation of non-rival knowledge in accordance with its marginal cost of production-zero will not provide the appropriate reward for the research effort that underlies the creation of new knowledge (see, for example, Barro and Sala-i-Martin, 2004). A decentralized framework with imperfect competition creates profit opportunities, which are the main drivers of technological change. Consequently in this paper we focus on the decentralized setting, which seems to be more appropriate for the analysis of technology-based economic growth.

4. Saddle-path stability of the balanced growth equilibrium

Once the uniqueness of the steady-state equilibrium has been established, this section is devoted to answering the main research question of this paper. When the economy is analyzed considering a research sector that is separate from the consumption-goods sector, we prove that a saddle-path stable steady state is possible. The instability problem associated with IRS to producible inputs in the research sector is overcome under certain conditions that define a feasible region for parameter values. These conditions seem to be neither unrealistic nor too restrictive.

Consumption, harvesting and labor share in the final output sector are decision variables. Thus, three variables in system (16)-(20) can be regarded as control variables (let us assume $c$, $u$ and $R/S$). Once these have been chosen, the two remaining variables ($y$ and $\chi$) are determined by the state variables $K$, $N$ and $S$. Therefore the stability of the steady-state equilibrium requires that the dimension of the stable manifold must be greater than or equal to two.

**Proposition 4.** Assuming $\alpha > \beta$ and

$$
\varepsilon \in \left[ 1 - \alpha, (1 - \alpha)(1 - \beta) \left( \frac{\alpha^2 + 1 - \beta}{\alpha^2 + 1 - \alpha - \beta} \right) \right], \quad (21)
$$
conditions 1, 2 and
\[ \eta \in \left( \frac{\alpha^2}{\alpha^2 + 1 - \alpha - \beta}, \frac{\alpha}{1 - \beta} \right), \] (22)
guarantee that the steady state is saddle-path stable. Specifically the stable manifold can be two or four dimensional.

**Proof.** The complete proof is presented in Appendix B, but here we present an outline.

The result is proved in two stages. First, it is proved that under conditions 1 and 2, necessary for the existence of the steady state, condition (22) is necessary and sufficient to ensure that the Jacobian matrix at the steady-state has a positive determinant. Therefore, it has either zero or an even number of negative eigenvalues.\(^{14}\) The second stage of the proof adds sufficient conditions \(\alpha > \beta\) and (21) to remove the case of no negative eigenvalues. The fifth column in the Jacobian matrix has all its elements equal to zero except for the entry in the main diagonal, which is therefore a (positive) eigenvalue. It is proved that the addition of the remaining elements in the main diagonal is negative. Therefore, the sum of the four remaining eigenvalues is negative, ensuring that there is at least one negative eigenvalue. ■

In the statement of Proposition 4, condition \(\alpha > \beta\), although not necessary, seems to agree with the empirical data (see Groth (2007) and references therein). Likewise, condition (21) is not necessary either. The standard model with an expanding variety of intermediate goods corresponds to \(\varepsilon = 1 - \alpha\), which satisfies condition (21). Thus, for this well-established model, the steady-state equilibrium is saddle-path stable under conditions 1, 2 and (22) together with \(\alpha > \beta\).

According to Proposition 4, for any set of initial conditions there exists at least one path converging toward the steady-state equilibrium. A two-dimensional stable manifold would guarantee that this path is unique, whereas indeterminacy would arise in case of a four-dimensional stable manifold. Indeterminacy means that from the same set of initial conditions there exists an infinite number of paths converging to the same steady

\(^{14}\)Henceforth a negative eigenvalue refers to an eigenvalue with a negative real part (either a complex or a real number).
state. However, for our model, numerical simulations show that the Jacobian matrix at
the steady state always has two negative eigenvalues. This result is robust to changes in
parameter values as long as conditions 1, 2 and (22) hold. It is also in accordance with
the recent findings in indeterminacy literature, which show that in closed economies it
is hard to satisfy the necessary conditions for generating indeterminacy.\footnote{For recent theoretical papers in this area, see, for example Meng and Velasco, 2004.}

\textit{The Social Planner’s Problem.} Regarding the stability for the centralized economy, in
Appendix~C we prove that under condition 2 the socially-optimum steady state does
not have the required saddle-path stability. Specifically, we prove that the Jacobian
matrix at the steady state has a unique negative eigenvalue, while a two-dimensional
stable manifold is needed. Again conditions 1 and 2 guarantee the existence of a positive
long-run growth rate with no need for an exogenous growth in population. However, the
equilibrium is saddle-path stable only if condition 2 is reversed. Reverting condition 2
would imply that a positive rate of economic growth would need population growth.

It is true then, as Groth and Schou (2002) anticipated, that without population
growth the equilibrium for the social planner’s problem remains unstable, even when the
R&D and the consumption-goods sectors are separately considered. Stability needs a
decentralized setting with imperfect competition, which acknowledges profit opportuni-
ties. The R&D sector arguably owes its existence to these profits, which constitute an
appropriate reward for the research efforts. It is in this framework that we prove the
saddle path stability of the equilibrium in the absence of population growth. Therefore
our model is fully endogenous, in contrast to the social planner’s problem, where the
requirement for population growth pushes this model towards semi-endogenous growth
literature.

\subsection*{4.1. Transitional adjustment}

We now turn to the transitional adjustment of the economy in response to some types
of disturbances. Because of the two-dimensional saddle-path stability of the steady state,
there always exists a unique policy choice for the control variables $c$, $u$ and $R/S$ which,
after an exogenous shock, guarantees a return path to the steady-state equilibrium. The numerical analysis presented in this section provides evidence of this stability.

Disturbances may produce imbalances between the different sectors. Starting from a steady-state position, new deposits of the exhaustible resource might be discovered, with the stock of capital and technology remaining unchanged. A war might destroy a fraction of the physical capital or the available deposits for the resource. Such shocks will cause the state variables to depart from their steady-state values.

Along the steady state the following ratios remain constant,

\[ E_1 = \frac{K^\eta}{N^{1-\varepsilon}}, \quad E_2 = \frac{S^\beta N^\varepsilon}{K^{1-\alpha}}. \]

Equations (9) and (10) establish the constancy of variables \( E_1 \) and \( E_2 \) at the steady state.\(^\text{16}\) The temporal evolution of these variables can be obtained from the system (16)-(20). The values of \( E_1 \) and \( E_2 \) at a given point in time are totally determined by the state variables \( K, N \) and \( S \), irrespective of the decisions taken at this time. By linearizing the dynamics of variables \( E_1, E_2, u, c \) and \( R/S \) in a neighborhood of the steady state, we can study the qualitative behaviour of the relevant variables along the transition.

The above method is used in our numerical example, where constants \( A, B \) and \( L \) have been normalized to 1; \( \rho = .05; \alpha = .4 \) seems an adequate value for developed countries (see Barro and Sala-i-Martin, 2004); according to Groth (2007), empirical data suggest that \( \alpha \) is several times the size of \( \beta \), hence we choose \( \beta = .15; \) empirical studies suggest consumers’ intertemporal elasticity of substitution, \( 1/\sigma \), lower than one (see, for example Farzin, 2004; and Groth, 2007). Hence we have chosen \( \sigma = 1.1 \). The standard case of a one-to-one technology function in the intermediate-goods sector is considered, that is, \( \varepsilon = 1 - \alpha; \) and finally \( \eta = .4 \) and \( \varphi = .7 \) are chosen so that conditions 1, 2 and (22) are fulfilled.

The existence of a two-dimensional stable manifold for the dynamic system in vari-

---

\(^{16}\)Variables \( E_1 \) and \( E_2 \) are obtained as a change of the original variables:

\[ E_1 = \frac{\chi}{B(1-v)^\eta[(1-u)L]^{1-\eta}}, \quad E_2 = \frac{y}{A^\alpha(\varphi L)^{1-\alpha-\beta}(\frac{\sigma}{\eta})^\beta}. \]
ables $E_1$, $E_2$, $u$, $c$ and $R/S$, means that stabilizing policy functions $u(E_1, E_2)$, $c(E_1, E_2)$ and $(R/S)(E_1, E_2)$, can be adopted. The graphs in Figure 2 show the contour lines of the surfaces $u(E_1, E_2)$, $c(E_1, E_2)$ and $(R/S)(E_1, E_2)$.

![Figure 2: Contour lines of the linearized stable manifold for $c$ (up-left), $u$ (up-right) and $R/S$ (down)](image)

The contour lines that mark lighter regions represent values above the steady state. Correspondingly, darker regions are linked with values below the steady-state value. The contour lines corresponding with the steady-state values $c^*$, $u^*$ and $(R/S)^*$ are highlighted. The linear shape of the contour lines responds to the linearization process that has been applied.

Starting from the steady-state position $P_0 = (E_1^*, E_2^*)$, a positive shock on the stock of the natural resource (new reservoirs are discovered), would push the ratio $E_2$ up to a higher level, with the ratio $E_1 = E_1^*$ remaining unchanged, to point $P_1$ in Figure 2. The portion of labor devoted to the production of output, $u$, should be initially lower than at
the steady state, \( u^* \). Correspondingly (because of the positive relation between \( u \) and \( v \) given in (15)), immediately after the shock, a lower portion of capital should be devoted to the sector of final output. These changes would initially imply more labor and capital devoted to the production of knowledge, and so, a higher knowledge growth rate, \( \chi > \chi^* \).

As Figure 2 (bottom) shows, the transition towards the steady state is only feasible if the depletion rate starts at \( P_1 \), below its steady-state value in \( P_0 \). However, this rate is numerically proven to be greater than the depletion rate immediately after the shock. This confirms that a more abundant stock of resource leads to greater harvesting. The increment in resource use counteracts the reduction in labor and capital, leading to a greater production of final output.

These policies will take the economy from \( P_1 \) to the long-run equilibrium, \( P_0 \), following the dashed lines in the direction indicated by arrows in Figure 2. The time paths of the relevant variables along the transition are depicted in Figure 3. The amount harvested initially and even the depletion rate after a short first interval are greater along the transition than at the steady state. Labor and capital shift from an initial period of excessive use in the research sector to a second period of excessive use in the consumption-goods sector. The main outcome after a positive shock on the stock of the resource is a greater growth rate of the economy along the transition, which enables a greater consumption per unit of capital.

A similar analysis could be carried out starting from any initial situation of unbal-
anced growth, possibly due to some exogenous shock.

5. Conclusions

We study a fully R&D-based endogenous growth model in which the consumption-goods and the knowledge creation sectors are separately considered. A non-renewable natural resource is, directly or indirectly, a productive input in both sectors. Our model qualifies as a hybrid model (in the terminology of Eicher and Turnovsky, 1999) because capital and knowledge enter the production function of final-good and technology. This assumption is equivalent to considering that although the resource directly enters the production of final output, it indirectly enters, through the manmade capital stock, the production function of innovation (the resource is growth-essential as stated by Groth, 2007). Because the use of the non-renewable natural resource has to diminish as time goes by, IRS could appear in the remaining inputs taken together.

The underlying R&D-based growth model is an economy with an expanding variety of intermediate goods. Throughout the paper the specification is maintained at its most general possible level, so the existence, the uniqueness and the saddle-path stability of a balanced growth path are proven for a wide range of situations.

The existence and uniqueness of a steady-state equilibrium with positive long-run growth rate needs two conditions. First, consumers must not be excessively reluctant to substitute present for future consumption. Second, returns to scale to producible inputs must be “globally” increasing. That is, if producible inputs show CRS in the final output sector, as in the standard model à la Romer, then IRS to producible inputs in the research sector are needed. Otherwise, when knowledge is a positive externality in the production of intermediate goods, the resulting IRS to producible inputs in the final output sector guarantee sustained growth even with slightly DRS to producible inputs in the research sector.

The assumption of a growth-essential resource implies a non-scaled balanced growth. Moreover, this equilibrium is not based on the knife-edge assumption of CRS to producible inputs, but it is feasible for a wide range of returns to scale to these inputs. More
importantly, the assumption of globally IRS to producible inputs does not necessarily lead to instability. Specifically, the existence of monopoly profits in the intermediate-goods sector and the assumption of a consumption-goods and an R&D sector separately considered, allows us to prove the saddle-path stability of the equilibrium. Contrary to what happened in one-sector models without market imperfections, stability does not necessarily require an exogenous growth in population. The saddle-path stability of the balanced growth path is proved, for the standard model of an expanding variety of intermediate goods à la Romer, as well as for a more general specification where knowledge spillovers also emerge in the intermediate-goods sector.

Long-run growth responds to policy instruments that affect the extraction rate. However it is policy invariant with respect to the conventional stimulus on savings or innovation. Although our model is fully endogenous, because of nonlinearities in the final output and the R&D sectors, it does not replicate the usual policy-dependency in fully-endogenous R&D-based growth models.

From any initial situation of unbalanced growth, we have numerically characterized the stabilizing policy functions that bring the economy toward the long-run equilibrium. Specifically, the transitional dynamics towards the steady-state equilibrium is studied after a positive shock on the stock of the resource, which drove the economy out of this equilibrium. We have obtained the time paths for the relevant variables along this transitory period. A transitory increment in the growth rate of the economy is observed.

The assumption of no population growth is made to highlight that the sustained growth is fully endogenous. The economy grows at a constant rate in the long run without the assistance of any exogenous source of growth. Nevertheless, the existence and stability of the balanced path could also be proved if population were assumed to grow at an exogenous constant rate, with the corresponding conditions upon parameters.

In our closed economy model, resource harvesting and research efforts take place within the same country. Empirical studies show that countries well endowed with natural resources often rely on technology developed abroad. Thus, a bilateral trade model between a country with a research sector better developed and a country rich in natural
resources would fit better with this empirical evidence. A matter for further research is the study of the existence of a stable sustained economic growth and the comparison of the growth rates for both economies with the growth rate under autarky. According to indeterminacy literature, in open economies indeterminacy arises more easily and, hence multiple equilibrium paths converging to the steady-state equilibrium might be feasible in the trade framework.

Appendix A: The decentralized economy

The exhaustible resource is equally shared by households who transfer their property rights to the extracting firms in exchange for the shares of these companies. In the same line as, for example, Scholz and Ziemes (1999) and Groth and Schou (2007), the resource is a commodity with perfectly well-defined property rights and the extraction, $R$, of the resource is carried out at no cost. Extracting companies supply the resource to output producers at a given market price, $p_R$. Thus, the benefit for harvesting firms can be written as:

$$
\int_{t}^{\infty} p_R(\tau) R(\tau) e^{-\bar{r}(\tau,t)(\tau-t)} d\tau,
$$

where $\bar{r}(\tau,t) = \left[1/(\tau-t)\right] \int_{t}^{\tau} r(s) ds$ is the average interest rate between times $t$ and $\tau$. The dynamics of the stock of the non-renewable resource, $S$, is given by (4). From the dynamic maximization problem of a representative extracting firm, the simplest Hotelling rule (6) follows.

In the final-goods sector, a representative manufacturer uses labor, $L_Y = uL$, intermediate inputs, $X(j)$, and the non-renewable resource, $R$, taking the available number of varieties of intermediate inputs, $N$, as given:\(^{17}\)

$$
Y = A \int_{0}^{N} (X(j))^{\alpha} L_Y^{1-\alpha-\beta} R^\beta dj, \ A > 0, \alpha, \beta > 0, \alpha + \beta < 1.
$$

\(^{17}\)Although the number of varieties of intermediate goods, $N$, is an integer, it is approximated here (as is often done in the literature, see, for example, Barro and Sala-i-Martin, 2004) by a continuous variable, obtaining essentially the same results.
Assuming perfect competition, instantaneous profit maximization in the final output sector leads to the following demand functions:

\[ \alpha A(X(j))^{\alpha - 1}(uL)^{1 - \alpha - \beta}R^\beta = p(j), \]
\[ (1 - \alpha - \beta)\frac{Y}{uL} = w, \]
\[ \beta \frac{Y}{R} = p_R, \]

where \( p(j) \) is the rental price of intermediate input \( j \) and \( w \) represents the wage paid to labor in the final-good sector.

The intermediate sector is composed of an infinite number of firms in the interval \([0, N]\). A particular firm has purchased a patented design from the R&D sector which gives it a monopoly power to produce its particular variety. Capital is rented to households at rate \( r \). Standard models with an increasing number of varieties usually assume that one unit of capital can be effortlessly transformed into a single unit of intermediate input according to a one-to-one production technology. A more general formulation could assume that the state of technology (identified as the number of varieties) increases the marginal productivity of the capital stock in the production of intermediate goods:

\[ X(j) = N^{\bar{\varepsilon}}K(j), \quad \bar{\varepsilon} \geq 0. \]

The particular case \( \bar{\varepsilon} = 0 \) occurs for the one-to-one production technology.

Because these firms are monopolists, they know the downward-sloping demand curves for their producer inputs, eq. (24). So, instantaneous benefits can be written as:

\[ \pi(j) = p(j)X(j) - rK(j) = \alpha A(X(j))^{\alpha}(uL)^{1 - \alpha - \beta}R^\beta - rN^{-\bar{\varepsilon}}X(j). \]

Profits maximization yields the following equations for price, quantity, and profit:

\[ p(j) = p = \frac{r}{\alpha}N^{-\bar{\varepsilon}} \quad \forall j \in [0, N], \]
\[ X(j) = X = \left( \frac{\alpha A(uL)^{1 - \alpha - \beta}R^\beta}{p} \right)^\frac{1}{1 - \alpha} \quad \forall j \in [0, N], \]
\[ \pi(j) = \pi = (1 - \alpha)pX = \alpha(1 - \alpha)\frac{Y}{N} \quad \forall j \in [0, N]. \]
By symmetry, each intermediate firm sets the same price and sells the same quantity of its producer input. As a consequence, and taking into account the production technology in (27), the aggregate capital used in the production of intermediates, $K_Y$, is given by:

$$K_Y = \int_0^N K(j) \, dj = X N^{1-\bar{\epsilon}}.$$  \hspace{1cm} (31)

Taking into account (31) into (23), the final output is given by expression (1), where\(^{18}\) $\bar{\epsilon} = 1 - \alpha (1 - \tilde{\epsilon})$. Furthermore, from equations (1), (28), (30) and (31) the relationship between the rate of return and the marginal productivity of the capital stock in the final output sector in (7) follows.

The monopolistic producer of an intermediate good must buy a patent from innovators. To purchase this patent he issues bonds, bought by households, who get an interest revenue determined by the rate of return. The decision to produce taken by intermediate firms depends on the difference between the cost of purchasing the patent from the R&D sector, $p_{RD}$, and the monopoly rents that can be obtained in exchange. Assuming free entry into this industry, the price for the patent must equal the ongoing profits of the monopolistic producer:

$$p_{RD}(t) = \int_t^\infty \pi(j)e^{-r(\tau,t)(\tau-t)} \, d\tau.$$  \hspace{1cm} (32)

The innovative sector is characterized by a number of perfectly competitive symmetric research firms. Following Jones (1995b), the contribution to knowledge creation of an R&D firm is equal to the number of researchers, $L_i$, times the rate at which R&D generates new ideas

$$\dot{N}_i = \delta L_i,$$

where $\tilde{\delta}$ is called the arrival rate. As proposed by Jones (1995b), this rate can be regarded as positively dependent on the amount of knowledge in the economy, $N$, and negatively dependent on the number of scientists, $L_i$. Furthermore, we assume that it also depends positively on the machinery used by researchers, $K_i$:

$$\tilde{\delta} = BK_i^\kappa L_i^{-\eta} N^{\varphi}, \quad \kappa, \eta, \varphi, B > 0.$$  

\(^{18}\) For $\bar{\epsilon} \leq 1$ then $\epsilon \in [1 - \alpha, 1]$, while for $\bar{\epsilon} > 1$ then $\epsilon > 1$.  

25
As in Jones (1995b), we consider an elasticity of the technical knowledge in the R&D sector lower than one, $\varphi < 1$.

An innovator creates new designs according to the following function:

$$\dot{N}_i = BK_i^\kappa L_i^{1-\eta}N^{\varphi}.$$

Under perfect competition, the production function must exhibit constant returns to scale with respect to rival inputs (capital and labor). That is $\kappa = \eta < 1$. Being symmetric, all firms take the same decisions when hiring capital and labor. Thus, the increment in the total number of varieties, $N = \sum_i N_i$, is given by (2), with $K_N = \sum_i K_i = (1 - v)K$ and $L_N = \sum_i L_i = (1 - u)L$.

A representative firm in the innovative sector maximizes instantaneous profits:

$$\pi_{RD} = p_{RD}BK_N^\eta L_N^{1-\eta}N^{\varphi} - wL_N - rK_N,$$

where $p_{RD}$ are the revenues from the patent of a specific intermediate good sold to the producer of this good. Assuming perfect competition, instantaneous profit maximization leads to the following demand functions

$$w = (1 - \eta)p_{RD}B((1 - v)K)^\eta(1 - u)L^{-\eta}N^{\varphi}, \quad (33)$$

$$r = \eta p_{RD}B((1 - v)K)^{\eta-1}(1 - u)L^{1-\eta}N^{\varphi}. \quad (34)$$

To close the model, households hold the capital stock and the bonds issued by producers of intermediate goods to buy patents. The households rent capital to producers of intermediate goods as well as to innovators, at the same rate of return. Thus households’ wealth reads:

$$V = p_{RD}N + K_Y + K_N.$$

The capital stock and the bonds are considered perfect substitute assets which yield the same return. In addition to these returns, households income is also increased by wages in the manufacturing and the R&D sectors. Besides, households own the extractive firm and get an equal share of the profits from harvesting. Because the economy is populated by a large number of identical individuals, the decision of a single agent regarding consumption
can be interpreted as the aggregate consumption, \( C \). Thus, the budget constraint for the households is:

\[
\dot{V} = rV + wL + p_R R - C, \quad V(0) = V_0. \tag{35}
\]

The households maximize their stream of discounted utility subject to their budget constraint:

\[
\max_{C} \int_{0}^{\infty} \frac{(C(t))^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt, \quad \text{s.t. (35)}.
\]

Positive parameter \( \sigma \) is the inverse of the elasticity of intertemporal substitution,\(^{19}\) and \( \rho \) is the constant rate at which utility flow is discounted over time.

Optimal consumers’ behaviour leads to the Ramsey rule (5). Finally, in a closed economy, the final output that is not consumed determines the investment in capital stock in (3).

**Appendix B**

**Derivation of the differential equations system (16)-(20)**

Equation (16) is obtained from equation (3), the Ramsey rule (5) and (7).

From (2) it immediately follows that

\[
\chi = \gamma_N = B ((1 - v)K)^{\eta} ((1 - u)L)^{1-\eta} N^{\sigma-1},
\]

taking into account (15) and differentiating, equation (18) follows.

Equation (1) can be rewritten as

\[ Y = AN \varepsilon (vK)^{\alpha} (uL)^{1-\alpha} R^{\beta}, \]

which taking log derivatives reads:

\[
\gamma_y = \gamma_Y - \gamma_K = \varepsilon \gamma_N + (\alpha - 1) \gamma_K + \left( \frac{\alpha - \varepsilon}{1 - \varepsilon} + 1 - \alpha - \beta \right) \gamma_u + \beta \gamma_R. \tag{36}
\]

From (26), the Hotelling rule in (6), and equation (7)

\[
\gamma_R = \gamma_Y - r = \gamma_Y - \beta \frac{\alpha^2 Y}{v R}, \tag{37}
\]

\(^{19}\)For \( \sigma = 1 \) a logarithmic utility function is considered.
Equation (19) stems from equations (36) and (37).

Differentiating (32)

$$\gamma_{pRD} = r - \pi \frac{\alpha^2 Y}{v K} - \alpha(1 - \alpha) \frac{Y}{N p_{RD}}.$$  \hspace{1cm} (38)

In addition, from (25) and (33)

$$p_{RD} = \frac{1 - \alpha - \beta}{1 - \eta} \frac{1 - u Y}{u N \chi}.$$ \hspace{1cm} (39)

Equations (38) and (39) lead to

$$\gamma_{pRD} = \frac{\alpha^2}{v} - (1 - \eta) \frac{\alpha(1 - \alpha)}{1 - \alpha - \beta} \chi.$$ \hspace{1cm} (40)

On the other hand, differentiating equation (39)

$$\gamma_{pRD} = -\frac{1}{1 - u} \gamma_u + \gamma_Y - \gamma_N - \gamma_\chi.$$ \hspace{1cm} (41)

Plugging (36) into (41) and equating to (40), equation (17) follows.

Finally, differentiating equation (26) and taking into account the Hotelling rule (6), we obtain

$$\gamma_R = \Omega(u)\gamma_u + \frac{\alpha}{1 - \beta} \left(1 - \frac{\alpha}{v} \right) y + \frac{\varepsilon}{1 - \beta} \chi - \frac{\alpha}{1 - \beta} c,$$

and equation (20) immediately follows.

**Proof of Proposition 3**

From (16) and (18), the steady-state values $c^*$ and $\chi^*$, can be written as functions of the steady-state value of variable $y$:

$$\chi^* = \frac{\eta}{\varphi - 1} \left(\frac{\rho}{\sigma} - \frac{\alpha^2}{\sigma v^*} y^*\right),$$  \hspace{1cm} (42)

$$c^* = \frac{\rho}{\sigma} - \left(\frac{\alpha^2}{\sigma v^*} - 1\right) y^*,$$ \hspace{1cm} (43)

where $v^*$ denotes the value of variable $v$ satisfying (15) when $u$ equates $u^*$. Plugging (42) and (43) in equation (19) the steady-state value of $y$ reads:

$$y^* = \frac{v^* - \rho}{\alpha^2 \bar{1} - \Lambda},$$ \hspace{1cm} (44)
where

\[ \Lambda = \frac{\sigma}{1 - \frac{(1-\alpha)(1-\varphi) - \epsilon \eta}{\beta(1-\varphi)}} \]  \hspace{1cm} (45) \]

Correspondingly, \( \chi^* \) and \( c^* \) can be obtained as functions of \( v^* \) and parameters.

Equating the term in brackets in equation (17) to zero, and rearranging terms it reads:

\[ -\frac{\alpha}{v^*} y^* + y^* - c^* - \frac{1-\beta}{\alpha} \eta (y^* - c^*) + \Gamma(u^*) \chi^* = 0. \]  \hspace{1cm} (46) \]

From expressions (42)-(44) it is immediately obvious that:

\[ \chi^* = \frac{\eta}{1-\varphi} (y^* - c^*), \]  \hspace{1cm} (47) \]

\[ y^* - c^* = \frac{\rho}{\sigma} \frac{\Lambda}{1-\Lambda}. \]  \hspace{1cm} (48) \]

Taking into account these two expressions, equation (46) can be rewritten as:

\[ -\frac{1}{\alpha} + \frac{\Lambda}{\sigma} \left\{ 1 - \frac{1-\beta}{\alpha} \eta \left( \frac{1-\beta - \epsilon}{\alpha} + \frac{(\varphi-1)(1-\beta)}{\alpha} \right) \left( \frac{1-\beta(1-\alpha)(1-\eta)}{1-\alpha - \beta} z^* \right) \right\} = 0 \]

with \( z^* = u^*/(1 - u^*) \). After some algebra it follows:

\[ z^* = -\frac{1}{\alpha \beta \eta(1-\alpha)(1-\eta)} \left[ (1-\alpha)(1-\varphi) - (\beta + \epsilon) \eta \right], \]  \hspace{1cm} (49) \]

which is positive under condition 1, implying a unique \( u^* \in (0,1) \) and \( v^* \in (0,1) \).

From conditions 1 and 2 it follows that \( y^* > 0 \) and \( \chi^* > 0 \). Furthermore, from (20) the steady-state value of \( R/S \) reads:

\[ \left( \frac{R}{S} \right)^* = \frac{\alpha}{1-\beta} c^* - \frac{\epsilon}{1-\beta} \chi^* - \frac{\alpha}{1-\beta} \left( \frac{1}{v} \right) y^*, \]

which is also positive under conditions 1 and 2.

Moreover, from (15), (16) and (44) it follows that \( c^* > 0 \) if and only if

\[ z^* > (1-\alpha - \beta) \frac{\eta}{1-\eta} \frac{\Lambda}{\sigma - \alpha^2 \Lambda} > 0, \]

or equivalently

\[ z^* > \frac{\eta(1-\alpha - \beta) \beta (1-\varphi)}{(1-\eta)(1-\beta(1-\varphi)(1+\alpha)(1-\alpha) - (1-\varphi)(1-\alpha) + \epsilon \eta)}. \]
Taking into account (49), after some algebra it can be easily proved that the above inequality is always satisfied under condition 1.

**Proof of Proposition 4**

**Stage 1**

From the system of differential equations (16)-(20), the Jacobian matrix in variables \( c, u, \chi, y, R/S \), at the steady state reads:

\[
J^* = 
\begin{pmatrix}
    a_{11} & a_{12} & 0 & a_{14} & 0 \\
    a_{21} & a_{22} & a_{23} & a_{24} & 0 \\
    a_{31} & a_{32} & a_{33} & a_{34} & 0 \\
    a_{41} & a_{42} & a_{43} & a_{44} & 0 \\
    a_{51} & a_{52} & a_{53} & a_{54} & a_{55}
\end{pmatrix}
\]

(50)

where

\[
a_{11} = c^*, \quad a_{12} = -\frac{\alpha}{\sigma} \Delta (u^*) y^* c^*, \quad a_{14} = \left( \frac{\alpha}{\sigma} \mu(u^*) - 1 \right) c^*,
\]

\[
a_{21} = u^* \Phi(u^*) \left( \frac{1}{\alpha} \eta - 1 \right),
\]

\[
a_{22} = u^* \Phi(u^*) \left( \Delta (u^*) y^* + \Gamma \frac{\chi^*}{(1-u^*)^2} \right),
\]

\[
a_{23} = u^* \Phi(u^*) \Gamma(u^*), \quad a_{24} = u^* \Phi(u^*) \left( 1 - \mu(u^*) - \frac{1}{\alpha} \eta \right),
\]

\[
a_{31} = \left( \frac{\Psi(u^*)}{u^*} a_{21} - \eta \right) \chi^*, \quad a_{32} = \frac{\Psi(u^*)}{u^*} a_{22} \chi^*,
\]

\[
a_{33} = \left( \frac{\Psi(u^*)}{u^*} a_{23} + \epsilon - 1 \right) \chi^*, \quad a_{34} = \left( \frac{\Psi(u^*)}{u^*} a_{24} + \eta \right) \chi^*,
\]

\[
a_{41} = \left( \frac{\Omega(u^*)}{u^*} a_{21} + \frac{1 - \alpha - \beta}{1 - \beta} \right) y^*, \quad a_{42} = \frac{\Omega(u^*)}{u^*} a_{22} + \frac{\alpha \beta}{1 - \beta} \Delta (u^*) y^* y^*,
\]

\[
a_{43} = \left( \frac{\Omega(u^*)}{u^*} a_{23} + \frac{\epsilon}{1 - \beta} \right) y^*, \quad a_{44} = \left( \frac{\Omega(u^*)}{u^*} a_{24} + \frac{\alpha \beta}{1 - \beta} \Delta (u^*) y^* \right) y^*,
\]

\[
a_{51} = \left( \frac{\Omega(u^*)}{u^*} a_{21} - \frac{\alpha}{1 - \beta} \right) \left( \frac{R}{S} \right)^*, \quad a_{52} = \frac{\Omega(u^*)}{u^*} a_{22} + \frac{\alpha}{1 - \beta} \Delta (u^*) y^* \left( \frac{R}{S} \right)^*,
\]

\[
a_{53} = \left( \frac{\Omega(u^*)}{u^*} a_{23} + \frac{\epsilon}{1 - \beta} \right) \left( \frac{R}{S} \right)^*, \quad a_{54} = \left( \frac{\Omega(u^*)}{u^*} a_{24} + \frac{\alpha \beta}{1 - \beta} \Delta (u^*) y^* \right) \left( \frac{R}{S} \right)^*,
\]

\[
a_{55} = \left( \frac{R}{S} \right)^*,
\]

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\[
\Delta(u^*) = \frac{\eta(1 - \alpha - \beta)}{\alpha(1 - \eta)} > 0 \quad \forall u^* \in (0, 1),
\]
\[
\mu(u^*) = \frac{\alpha}{v^*} = \alpha + \frac{1 - \alpha - \beta - \eta}{\alpha} \frac{1 - u^*}{u^*},
\]
\[
\Upsilon = \frac{(1 - \eta)(1 - \beta)(1 - \alpha)}{1 - \alpha - \beta}.
\]

At the steady state, \((R/S)^* > 0\), hence the sign of \(\text{det } J^*\) equates the sign of the 4th leading principal minor. Further, \(c^*, u^*, \chi^*, y^* > 0\), so after some simplifications which do not change the sign of the determinant\(^{20}\) \((f_1/c^*, f_2/u^*, f_3/\chi^* - \Psi(u^*)f_2/u^*, f_4/y^* - \Omega(u^*)f_2/u^*)\), this principal minor equals

\[
\Phi(u^*) \begin{vmatrix}
1 & -\frac{\alpha}{v^*} \Delta(u^*)y^* & 0 & \frac{\alpha^2}{v^*} - 1 \\
\frac{1 - \beta}{\alpha \eta} - 1 & \frac{1 - \beta}{\alpha \eta} \Delta(u^*)y^* + \Upsilon \frac{\chi^*}{(1 - u^*)^2} & \Gamma(u^*) & 1 - \frac{\alpha}{v^*} - \frac{1 - \beta}{\alpha \eta} \\
-\eta & 0 & \varphi - 1 & \eta \\
\frac{1 - \alpha - \beta}{1 - \beta} & \frac{\alpha^2}{1 - \beta} \Delta(u^*)y^* & \frac{\alpha^2}{1 - \beta} & \frac{\alpha^2}{1 - \beta} (1 - \varphi) - (1 - \alpha)
\end{vmatrix}.
\]

Expanding this determinant along the second column, it follows:

\[
\Phi(u^*) \left( \Delta(u^*)y^* \Theta + \Upsilon \frac{\chi^*}{(1 - u^*)^2} \Xi \right),
\]

where

\[
\Theta = \begin{vmatrix}
1 & -\frac{\alpha}{v^*} & 0 & \frac{\alpha^2}{v^*} - 1 \\
\frac{1 - \beta}{\alpha \eta} - 1 & \frac{1 - \beta}{\alpha \eta} & \Gamma(u^*) & 1 - \frac{\alpha}{v^*} - \frac{1 - \beta}{\alpha \eta} \\
-\eta & 0 & \varphi - 1 & \eta \\
\frac{1 - \alpha - \beta}{1 - \beta} & \frac{\alpha^2}{1 - \beta} & \frac{\alpha^2}{1 - \beta} (1 - \varphi) - (1 - \alpha)
\end{vmatrix}
\]

and

\[
\Xi = \begin{vmatrix}
1 & 0 & 0 & \frac{\alpha^2}{v^*} - 1 \\
\frac{1 - \beta}{\alpha \eta} - 1 & \frac{1 - \beta}{\alpha \eta} & \Gamma(u^*) & 1 - \frac{\alpha}{v^*} \\
-\eta & 0 & \varphi - 1 & \eta \\
\frac{1 - \alpha - \beta}{1 - \beta} & 0 & \frac{\alpha^2}{1 - \beta} & \frac{\alpha^2}{1 - \beta} (1 - \varphi) - (1 - \alpha)
\end{vmatrix}.
\]

---

\(^{20}\) Here \(f_i\) denotes the \(i\)-th file.
Adding the first and fourth columns of $\Theta$, it immediately follows that this determinant is zero, because it has two proportional columns.

For $\Xi$, after some algebra,
\[ \Xi = 1 - \varphi \frac{\alpha^2 \beta}{1 - \beta} \sigma v^* \left[ \sigma - 1 + \frac{(1 - \alpha)(1 - \varphi) - \varepsilon \eta}{\beta(1 - \varphi)} \right]. \]

Under condition 2 and because $\varphi < 1$, it follows that $\Xi < 0$.

Because at the steady state $\chi^*, y^*, (1 - u^*), \Delta(u^*) > 0$, from (51) it follows that the sign of $\det J^*$ equals the sign of $-\Phi(u^*)$, which is positive if and only if $[\eta - \alpha/(1 - \beta)](u^* - v^*) < 0$.

From (15), it follows:
\[ u^* > v^* \iff \eta > \frac{\alpha^2}{\alpha^2 + 1 - \alpha - \beta}. \]
And it is easy to see that:
\[ \frac{\alpha^2}{\alpha^2 + (1 - \alpha - \beta)} < \frac{\alpha}{1 - \beta} < 1. \]
Therefore under condition (22), $\Phi(u^*) < 0$ and $\det J^* > 0$. Hence, the number of negative eigenvalues is either even or null.

**Stage 2**

Taking into account expressions (47) and (48),
\[ a_{33} = \frac{\alpha y^*}{v^*} \left\{ \Psi(u^*)\Phi(u^*) \left(1 - \frac{\alpha}{\sigma} \left(1 - \frac{1 - \beta}{\alpha} \eta\right)\right) - \frac{\eta \alpha}{\sigma} \right\} \]
\[ + \frac{\rho}{\sigma} \left(\eta + \Psi(u^*)\Phi(u^*) \left(1 - \frac{1 - \beta}{\alpha} \eta\right)\right). \]

From (42)
\[ a_{22} = \Phi(u^*)u^* \left\{ \frac{\eta(1 - \alpha - \beta)}{\alpha(1 - \eta)} \frac{y^*}{(u^*)^2} - \frac{(1 - \eta)(1 - \beta)(1 - \alpha)\eta}{(1 - \alpha - \beta)(1 - \varphi)\sigma(1 - u^*)^2} \left(\rho - \frac{\alpha^2 y^*}{v^*}\right) \right\}. \]

Replacing $y^*$ by its expression in (44), the addition of the first four elements in the main diagonal, $\Sigma = a_{11} + a_{22} + a_{33} + a_{44}$, can be written as:
\[ \frac{\rho}{1 - \Lambda} \left\{ \frac{-\Lambda}{\sigma} - \frac{v^*}{\alpha^2} \frac{1 - u^*}{u^* - v^*} - \frac{\beta}{1 - \beta} \frac{u^*}{\sigma u^* - v^*} - \frac{\Omega(u^*) - \Psi(u^*)}{(1 - \beta)\eta - \alpha} \frac{1 - u^*}{u^* - v^*} \right\} \]
\[ + \frac{\alpha \eta}{(1 - \beta)\eta - \alpha u^* - v^*} \left\{ \frac{v^*}{\alpha^3} \frac{1 - \alpha - \beta}{1 - \eta} \frac{1 - u^*}{u^*} + \frac{(1 - \eta)(1 - \beta)(1 - \alpha)\Lambda}{(1 - \alpha - \beta)(1 - \varphi)\sigma} \frac{1 - u^*}{1 - u^*} \right\}. \]
Because \( \Lambda < 1 \), the sign of \( \Sigma \) coincides with the sign of the term in brackets.

Rearranging terms and taking into account that:

\[
\frac{\Omega(u^*) - \Psi(u^*)}{(1 - \beta)\eta - \alpha} = \frac{\alpha}{1 - \beta \eta - \alpha} \frac{1}{u^* - v^*} - \frac{\alpha}{1 - \beta},
\]

the term in brackets above can be rewritten as:

\[
\frac{1}{u^* - v^*} \left\{ \frac{v^*(1 - u^*)}{\alpha^2} - \frac{\Lambda u^*}{\sigma} + \frac{\alpha}{(1 - \beta)\eta - \alpha} \left[ -\frac{1}{\alpha} + \frac{v^*}{\alpha^3} \eta(1 - \alpha - \beta) \frac{1 - u^*}{1 - \eta} \right] 
+ \frac{\eta(1 - \eta)(1 - \beta)(1 - \alpha)\Lambda}{(1 - \alpha - \beta)(1 - \varphi)\sigma} \frac{u^*}{1 - u^*} \right\} + 1 - \frac{\Lambda}{\sigma},
\]

From (15) and (22) it follows that \( u^* - v^* > 0 \). Therefore, the sign of the expression above is not modified when multiplied by \( u^* - v^* \). Furthermore, replacing \( v^* \) using expression (15), and rearranging terms, the sign of \( \Sigma \) is given by the sign of:

\[
-\frac{1 - u^*}{\alpha^2 + (1 - \alpha - \beta) \frac{\eta}{1 - \eta} \frac{1 - u^*}{u^*}} \Sigma_1 - \frac{\Lambda}{\sigma} u^* - \frac{\alpha}{1 - \beta \eta - \alpha} \Sigma_2,
\]

where

\[
\Sigma_1 = 1 - \left( (1 - \alpha - \beta) \frac{\eta}{1 - \eta} - \alpha \right) \left( 1 - \frac{\Lambda}{\sigma} \right),
\]

\[
\Sigma_2 = \frac{\alpha}{\alpha^2 + (1 - \alpha - \beta) \frac{\eta}{1 - \eta} \frac{1 - u^*}{u^*}} - \frac{\Lambda}{\sigma} \frac{(1 - \beta)(\beta \eta - (1 - \alpha)(1 - \varphi - \eta))}{\alpha \beta (1 - \varphi)}.
\]

Replacing \( \Lambda \) by its expression in (45), after some manipulations, it follows that the sign of \( \Sigma_1 \) is the sign of the expression below:

\[
1 + \alpha^2 - (1 - \alpha - \beta) \frac{\eta}{1 - \eta} - \frac{\beta (1 - \varphi)}{(1 - \alpha)(1 - \varphi) - \varepsilon \eta}.
\]

Under condition (22), the expression above and hence \( \Sigma_1 \) are positive.

Replacing \( \Lambda \) again by its expression, after some simplifications, the sign of \( \Sigma_2 \) coincides with the sign of:

\[
\alpha [\alpha (1 - \varphi) + \eta (\varepsilon - 1)] + \eta [\alpha \beta - (1 - \beta)(1 - \alpha)\eta].
\]

Because \( 1 - \varphi - \varepsilon \eta/(1 - \alpha) < 0 \), then

\[
\alpha (1 - \varphi) + \eta (\varepsilon - 1) < \eta \left( \frac{\varepsilon}{1 - \alpha} - 1 \right).
\]
Therefore, a sufficient condition for $\Sigma_2 \leq 0$ is

$$\alpha \left( \frac{\varepsilon}{1 - \alpha} - 1 \right) + \alpha \beta - (1 - \beta)(1 - \alpha) \eta \leq 0. \quad (52)$$

Under condition (22), $\eta > \alpha^2/(\alpha^2 + 1 - \alpha - \beta)$, and a sufficient condition for the fulfilment of inequality (52) is:

$$\alpha \left( \frac{\varepsilon}{1 - \alpha} - 1 \right) + \alpha \beta - (1 - \beta)(1 - \alpha) \frac{\alpha^2}{\alpha^2 + 1 - \alpha - \beta} \leq 0.$$  

After some calculus, the above inequality can be rewritten as:

$$(\alpha^2 + 1 - \alpha - \beta)(\varepsilon - (1 - \alpha)(1 - \beta)) - (1 - \alpha)(1 - \beta)\alpha \leq 0.$$  

From this inequality the following upper bound for $\varepsilon$ can be derived:

$$1 - \alpha \leq \varepsilon \leq (1 - \alpha)(1 - \beta) \left( \frac{\alpha^2 + 1 - \beta}{\alpha^2 + 1 - \alpha - \beta} \right). \quad (53)$$

The range of values for $\varepsilon$ in (53) is not empty if and only if:

$$(\alpha^2 + 1 - \beta)\beta - \alpha < 0.$$  

The above condition applies if and only if

$$\frac{1 - \sqrt{1 - 4\beta^2(1 - \beta)}}{2\beta} < \alpha < 1.$$  

It can be easily proved that $\alpha > \beta$ ensures that the lower bound in inequality above is satisfied. Therefore, $\alpha > \beta$ and (53) are sufficient conditions for $\Sigma_2 < 0$ and consequently for $\Sigma < 0$. Therefore, the Jacobian matrix evaluated at the steady state presents at least one negative eigenvalue.

**Appendix C: The social planner problem**

The social planner maximizes the utility of a representative consumer, and solves the optimization problem:

$$\max_{u, v, c, R} \int_0^\infty \frac{e^{1-\sigma} - 1}{1 - \sigma} e^{-\mu t} dt$$
subject to constraints (1)-(4). Setting up the usual Hamiltonian, first-order conditions for an interior solution are

\[
\begin{align*}
 c^{-\sigma} &= \lambda, \\
 \frac{Y}{v} &= p\eta \frac{\dot{N}}{1-v}, \\
 (1 - \alpha - \beta) \frac{Y}{u} &= p(1-\eta) \frac{\dot{N}}{(1-u)}, \\
 \beta \frac{Y}{R} &= q, \\
 \frac{\dot{\lambda}}{\lambda} &= \rho - \alpha \frac{Y}{vK}, \\
 \frac{\dot{p}}{p} &= \alpha \frac{Y}{vK} - \left( \varepsilon(1-\eta) \frac{u}{1-\alpha - \beta 1 - u} + \varphi \right) \frac{\dot{N}}{N}, \\
 \frac{\dot{q}}{q} &= \alpha \frac{Y}{vK},
\end{align*}
\]

(54) (55)

where \(\lambda\) is the shadow price of capital, while \(p\) and \(q\) are the shadow prices of knowledge and natural resource in units of capital.

From (54) and (55) follows the optimal relationship between the share of labor and the share of capital in the final output sector:

\[
\frac{\alpha}{1-\alpha - \beta} \frac{1-v}{v} = \frac{\eta}{1-\eta} \frac{1-u}{u},
\]

which differs from its corresponding relation (15) in the decentralized setting. Proceeding as in the derivation of system (16)-(20) in Appendix B, the following system of differential equations is obtained:

\[
\begin{align*}
 \gamma_c &= \left( \frac{\alpha}{\sigma v} - 1 \right) y + c - \frac{\rho}{\sigma}, \\
 \gamma_u &= \Phi(u) \left\{ \left( 1 - \frac{1}{v} - \frac{1 - \beta}{\alpha} \eta \right) y + \left( \frac{1 - \beta}{\alpha} \eta - 1 \right) c + \Gamma(u) \chi \right\}, \\
 \gamma_x &= \Psi(u) \gamma_u + \eta y - \eta c + (\varphi - 1)X, \\
 \gamma_y &= \Omega(u) \gamma_u + \left\{ \alpha - 1 - \frac{\alpha \beta}{1 - \beta} \left( \frac{1}{v} - 1 \right) \right\} y + \frac{1 - \alpha - \beta}{1 - \beta} c + \frac{\varepsilon}{1 - \beta} \chi, \\
 \gamma_R &= \Omega(u) \gamma_u + \frac{\alpha}{1 - \beta} \left( 1 - \frac{1}{v} \right) y + \frac{\varepsilon}{1 - \beta} \chi + \frac{R}{S} - \frac{\alpha}{1 - \beta} c,
\end{align*}
\]

(56) (57) (58) (59) (60)

where

\[
\Gamma(u) = \frac{\varepsilon}{\alpha} \left( 1 + \frac{(1-\beta)(1-\eta)}{1-\alpha-\beta} \frac{u}{1-u} \right).
\]

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Proposition 5. Under conditions 1 and 2, there exists a unique steady-state equilibrium for the social planner economy with \( u_{sp}^* \in (0, 1) \) and \( c_{sp}^* \), \( y_{sp}^* \), \( \chi_{sp}^* \), \( (R/S)_{sp}^* > 0 \), where the subscript \( sp \) stands for social planner.

Proof. Following the same procedure as in the proof of Proposition 3, the steady-state equilibrium for the social planner economy is given by:

\[
\begin{align*}
\chi_{sp}^* &= \frac{\eta}{1 - \varphi} (y_{sp}^* - c_{sp}^*), \quad y_{sp}^* - c_{sp}^* = \frac{\rho}{\sigma} \frac{\Lambda}{1 - \Lambda} \quad y_{sp}^* = \frac{v_{sp}^*}{\alpha} \frac{\rho}{1 - \Lambda},
\end{align*}
\]

and \( z_{sp}^* = u_{sp}^*/(1 - u_{sp}^*) \) is:

\[
\begin{align*}
z_{sp}^* &= -\frac{1 - \alpha - \beta}{\varepsilon \beta \eta (1 - \eta)} [(1 - \alpha)(1 - \varphi) - (\beta(1 - \varphi) + \varepsilon) \eta],
\end{align*}
\]

which is positive under condition 1, implying a unique \( u_{sp}^* \in (0, 1) \).

Following the same argumentation as in the proof of Proposition 3, the existence and uniqueness of the steady state are proved.

Differences between \( z_{sp}^* \) in (61) and its corresponding value in the decentralized economy (49) imply differences in the steady-state values of control and state variables. Nevertheless, the reasoning followed to derive equations (11) and (13) remains the same in the social planner problem. Therefore, the growth rate in the decentralized economy coincides with the growth rate in the optimally planned economy, if conditions 1 and 2 are still satisfied.

A stability analysis of the steady state requires the study of the Jacobian matrix of system (56)-(60) at the steady state, \( J_{sp}^* \), which differs from matrix (50) in two aspects. First, motivated by internalizing the positive effects of knowledge on the output production, the definition of constant \( \bar{\Upsilon} \) in element \( a_{22} \), now for matrix \( J_{sp}^* \) reads:

\[
\bar{\Upsilon} = \frac{(1 - \beta)(1 - \eta)\varepsilon}{\alpha(1 - \alpha - \beta)}.
\]

The second difference is the definition of function \( \bar{\mu}(u) \), due to the absence of monopoly benefits in the social planner problem, this is now given by

\[
\bar{\mu}(u_{sp}^*) = \frac{1}{v_{sp}^*} = 1 + (1 - \alpha - \beta) \frac{\eta}{1 - \eta} \frac{1 - u_{sp}^*}{u_{sp}^*}.
\]

Given this Jacobian matrix, the following lemma is proved.
Lemma 6. The sign of $\det J^*_{sp}$ is negative if and only if condition 2 is satisfied.

Proof. Following the same reasoning as in the proof of proposition 4 (see Appendix B), we can assert that the sign of $\det J^*_{sp}$ is the same as the sign of the following expression

$$\Phi(u^*_{sp}) \left( \Delta(u^*_{sp}) y^*_{sp} \hat{\Theta} + \frac{\lambda^*_{sp}}{(1 - u^*_{sp})^2} \hat{\Xi} \right),$$

where $\hat{\Theta} = 0$, as before in (51),

$$\Phi(u^*_{ps}) = \frac{\alpha (1 - u^*_{sp})}{[(1 - \beta) \eta - \alpha] (u^*_{sp} - v^*_{sp})} = \frac{\alpha (1 - \eta) + (1 - \alpha - \beta) \eta \frac{1 - u^*_{sp}}{u^*_{sp}}}{[(1 - \beta) \eta - \alpha] u^*_{sp}} > 0,$$

and under condition 2

$$\hat{\Xi} = \frac{1 - \varphi}{1 - \beta} \frac{\alpha \beta}{\sigma v^*_{sp}} \left[ \sigma - 1 + \frac{(1 - \alpha)(1 - \varphi) - \varepsilon \eta}{\beta (1 - \varphi)} \right] < 0.$$

Therefore, as proved in Martínez-García (2003), if $\det J^*_{sp}$ is negative, the socially optimal steady state in an endogenous growth model with three state variables has a one-dimensional stable manifold. The matrix has only one negative eigenvalue, while stability would require two negative eigenvalues.

References