Extensions of Dagum’s Gini decomposition

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Summary: The purpose of this paper is to extend Dagum’s Gini decomposition (“A New Approach to the Decomposition of the Gini Income Inequality Ratio”, Empirical Economics 22(4), 515-531, 1997a) following three types of theoretical modelisations. The first one deals with a “poor/non-poor” decomposition within a sub-group multilevel framework. The second one exhibits a multi-decomposition technique, that is, the combination of the multilevel sub-group decomposition and the income source decomposition. Finally, we provide a parametric multi-decomposition in order to capture different dimensions of income inequality within groups and between groups.

Keywords: Gini, Income Source Decomposition, Multi-decomposition, Poverty, Sub-group Decomposition.

1. Introduction

For a long time the Gini decomposition has been considered as a non-appealing method compared with the measures derived from the second law of entropy. Nevertheless, many authors have continued to develop new Gini decompositions, which [following Shorrocks (1999)] exercised their mind. The first one can be attributed to Soltow (1960). Then, Bhattacharya and Mahalanobis (1967) introduced a Gini decomposition on the basis of Gini’s mean difference index (1912) in order to study regional disparities. Afterwards, Rao (1969) first proposed the two existing procedures to

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decompose an income inequality measure, that is: the sub-group (subpopulation) decomposition; and, the income source decomposition (factor component). His sub-group decomposition is an atypical one. He offers a two-term Gini decomposition with a quadratic form, which points out the fundamental difference that occurs between the Gini coefficient and the entropy measures. The Gini ratio provides an unusual between-group component because it measures the income inequalities between each and every pair of sub-populations, whereas entropy and most of between-group inequality measures yield the income inequalities in mean between the sub-populations. Rao’s decomposition by factor component makes possible the computation of each income source contribution (such as wages, fringe benefits, child support benefits, transfers, etc.) to the amount of the overall inequality.

Mookherjee and Shorrocks (1982) demonstrated that Gini index can be rewritten in order to bring out a between-group Gini index in mean plus a residual term, say “interaction term”. Their conclusion has incited many researchers to reject the Gini coefficient as a good decomposable measure. Contrary to this, the entropic measures were increasingly used, generalized [see Bourguignon (1979), Shorrocks (1980), Cowell (1980a, 1980b)], and defined as sub-group consistent.

The sub-group consistency is a new decomposition property introduced by Shorrocks (1984, 1988) in the field of income inequality measurement and particularly via the entropic indices. Let us remember. Imagine a change in incomes of group $A$, ceteris paribus, such that mean incomes and population shares remain unchanged. If the inequality is increasing (decreasing) in group $A$, then the overall inequality is (not strictly) increasing (decreasing). This property relies on the notion of sub-group monotonicity, introduced by Foster, Greer and Thobercke (1984).

A problem arises with the sub-group consistency property since it leads inevitably to a between-group component, which represents the inequalities in mean between the groups. In this sense, Dagum (1997a) advanced some critical insights. Indeed, this kind of measures needs an equally distributed income vector within each group that is equal to the mean income of the corresponding group. Then, inequalities in mean between the groups are not valid and cannot characterize a good statistical measure since it is very similar to the one-way variance analysis (ANOVA), for which: (i) the sub-groups have equal variances; (ii) the observations are statistically independent; and (iii) the distributions are equally distributed.

Following these statistical limitations concerning the sub-group consistency property, we investigate new kinds of decomposition using the Gini coefficient and extending Dagum’s Gini decomposition (1997a, 1997b). After reviewing Dagum’s approach (Section 2), we deal with our initial model, the multilevel Gini decomposition, that is, a sequence of overlapping sub-group decompositions with the case of “poor/non-poor” partitions
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(Section 3). The second modelisation exposes a mixture of decompositions, the so-called Gini multi-decomposition, which merges the sub-group decomposition with the income source decomposition (Section 4). This technique is apprehended within a multidimensional framework and then combined with the multilevel Gini decomposition to yield our third model (Section 5). Finally, Section 6 is devoted to the concluding remarks.

2. Dagum’s Gini decomposition by sub-population (1997a, 1997b)

Let \( P \) be a population with \( n \) income units: \( x_i (i = 1, \ldots, n) \). \( P \) is partitioned into \( k \) sub-populations \( P_j \) \((j=1, \ldots, k)\) of size \( n_j \) with mean income \( \mu_j \). Let \( \mu \) be the mean income of \( P \). The income level of the \( i \)-th individual that belongs to the \( j \)-th group is \( x_{ij} (i = 1, \ldots, n_j) \). Then, the overall income vector of \( P \) can be written as:

\[
\left([x_1, \ldots, x_{i}, \ldots, x_n]\right) = \left([x_{i_1}, \ldots, x_{n_1}], \ldots, [x_{i_j}, \ldots, x_{n_j}], \ldots, [x_{i_k}, \ldots, x_{n_k}]\right)
\]

The Gini index, based on Gini’s mean difference (1912), is given by:

\[
G = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j|}{2nn\mu}.
\]

**Definition 1.** The Gini index associated with sub-group \( P_j \) yields the income inequality within \( P_j \):

\[
G_{jj} = \frac{\sum_{i=1}^{n_j} \sum_{j=1}^{n_j} |x_{ij} - x_{ij}|}{2n_jn_j\mu_j}.
\]

**Definition 2.** The Gini index associated with two sub-groups \( P_j \) and \( P_h \) [Dagum (1987)] quantifies the income inequalities between \( P_j \) and \( P_h \):

\[
G_{jh} = \frac{\sum_{i=1}^{n_j} \sum_{h=1}^{n_h} |x_{ij} - x_{ih}|}{(\mu_j + \mu_h) n_j n_h}, \forall j, h = 1, \ldots, k.
\]

The \( G_{jh} \) index involves \( n_j \times n_h \) binary income differences between \( P_j \) and \( P_h \), whereas \( G_{jj} \) involves \( n_j \times n_j \) binary income differences within \( P_j \). Gathering
these income differences within groups and between groups, we obtain Dagum’s two-term Gini decomposition by sub-population:

\[ G = \frac{\sum_{j=1}^{k} \left( \frac{n_j}{n} \sum_{i=1}^{n_j} x_{ij} - x_{rh} \right)}{2\mu n n} + \frac{2}{2\mu n n} \sum_{j=2}^{k} \left( \frac{n_j}{n} \sum_{i=1}^{n_j} x_{ij} - x_{rh} \right) \]

\( (5) \)

\[ G = G^w + G^{gb}. \]

\( (6) \)

\( G^w \) characterizes the within-group Gini index, and \( G^{gb} \) represents the gross between-group Gini index that gauges the inequalities between each and every pair of sub-groups [in the same way as Rao (1969)]. The expression “gross between-group” singles out \( G^{gb} \) from the traditional between-group measures, which are only based on mean incomes.

The Gini index can be formulated as a weighted average of the Gini indices associated with each group \( G_{j j} \), and the Gini indices associated with pairs of groups \( G_{j h} \). The weights are population shares and income shares, respectively:

\[ p_j = \frac{n_j}{n}, \quad s_j = \frac{n_j \mu_j}{n \mu}. \]

\( (7) \)

Then, the Gini index is:

\[ G = \sum_{j=1}^{k} G_{j j} p_j s_j + \sum_{j=2}^{k} \left( \frac{1}{2} \sum_{j=1}^{k} G_{j h} \left( p_j s_j + p_j s_j \right) \right) \]

\[ = G^w + G^{gb}. \]

\( (8) \)

Let us introduce the gross economic affluence and the first-order moment of transvariation in order to implement a decomposition in three components [see also Silber (1989), Lerman and Yitzhaki (1991), etc.].

**Lemma 1 [Dagum (1980)].** The gross economic affluence \( d_{jh} \) is the weighted average of the income difference \( x_{ij} - x_{rh} \) for all incomes \( x_{ij} \) of the members of \( P_j \) with income greater than \( x_{rh} \) of the members of \( P_h \) such that \( P_j \) is more “affluent” than \( P_h \) (\( \mu_j > \mu_h \)):

\[ d_{jh} = \int_0^x d F_j(x) \frac{x}{0} (x - y) d F_h(y). \]

\( (9) \)
**Lemma 2 [Dagum (1997a)].** The first-order moment of transvariation $p_{jh}$ between the $j$-th and the $h$-th sub-populations (such as $\mu_j > \mu_h$) is:

$$p_{jh} = \int_0^\infty dF_h(x) \int_0^x (x-y) dF_j(y).$$  

(10)

The transvariation [Gini (1916), Dagum (1959, 1960, 1961)] stands to the fact that income differences are of opposite sign compared with the difference of their corresponding mean incomes.

The gross economic affluence and the first-order moment of transvariation yield the relative economic affluence (directional economic distance ratio, “economic distance” from now on) between the $j$-th and the $h$-th sub-populations:

$$D_{jh} = \frac{d_{jh} - p_{jh}}{d_{jh} + p_{jh}}.$$  

(11)

$D_{jh}$ is included in the close interval $[0,1]$. $D_{jh} = 0$ iff $\mu_j = \mu_h$, and $D_{jh} = 1$ iff the probability distribution functions of $P_j$ and $P_h$ do not overlap. It is a normalized measure of dimension zero since $d_{jh}$ and $p_{jh}$ have the same dimension. The relative economic affluence permits the gross between-group inequality to be separated in two contributions:

- the net contribution of the extended Gini between groups to the overall Gini index:

$$G_{nb} = \sum_{j=2}^k \sum_{h=1}^{j-1} G_{jh} D_{jh} (p_j s_a + p_h s_j);$$  

(12)

- and the contribution of the intensity of transvariation between groups to the overall Gini index

$$G' = \sum_{j=2h=1}^{k} G_{jh} (1-D_{jh}) (p_j s_a + p_h s_j).$$  

(13)

Indeed, the gross between-group Gini index can be expressed as follows:

$$G_{gb} = G_{nb} + G'.$$  

(14)

The transvariation intensity measures the inequalities of overlap between income distributions, that is, the inequalities inherent to the high incomes of the poorest sub-populations (in mean). The net between-group Gini measures
the inequalities of non-overlap between the income distributions, that is, those generated by the high incomes of the richest sub-groups (in mean) as well as the inequalities in mean (since $G_{nb}$ depends on $D_{jh}$, where $D_{jh} = 0$ iff $\mu_j = \mu_h$).

**Theorem [Dagum (1997a)]**. The Gini coefficient computed on $P$ of size $n$, partitioned into $k$ sub-groups of size $n_j$ ($j = 1, \ldots, k$) is given by:

$$G = G^w + G^{ab} + G^t.$$  

(15)

3. The multilevel Gini decomposition

3.1 The general framework

Cowell (1985), Wodon (1999) and Salas (2002) introduced the multilevel decomposition technique. The overall population is divided into a first partition $K$, with $k$ groups $P_j$ of size $n_j$ ($j = 1, \ldots, k$). Let $S$ be a second partition (or equivalently the sub-partition of $K$). Suppose that each group $P_j$ is separated into $q_j^2 \leq n_j$ sub-groups (hereafter the groups are associated with the first partition $K$ and the sub-groups with the second partition $S$).\(^1\)

Salas’s result about the multilevel entropy decomposition is the following:

$$I = I^{b.K} + I^{b.SK} + I^{w.S},$$  

(16)

where $I^{b.K}$ is the between-group inequality (in mean) among the groups of the first partition $K$; $I^{b.SK}$ is the between-group measure (in mean) along the sub-groups of the second partition $S$; and, $I^{w.S}$ is the inequalities within the sub-groups of $S$.

Let us extend Dagum’s Gini decomposition and introduce the multilevel framework. We name $n_z$ and $\mu_z$ (and $n_w$, $\mu_w$) the number of individuals and the mean income of the $z$-th sub-group ($w$-th) of the second partition, respectively ($z, w = 1, \ldots, q_j$), $z$ and $w$ being the sub-groups of group $P_j$ of the first partition. Therefore, we can calculate the inequalities within the $z$-th sub-group of the $j$-th group:

\(^1\) Every superscript 2 in this paper refers to the second partition $S$, e.g. $q_j^2$ is the number of groups in the second partition, that is, each $P_j$ has $q_j^2$ groups.
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\[
G_{j,zz}^2 = \frac{\sum_{i=1}^{n_z} \sum_{r=1}^{n_z} |x_{iz} - x_{riz}|}{2n_z n_z \mu_z},
\]

(17)

and the inequalities between the \(z\)-th and the \(w\)-th sub-groups of \(P_j\) of the first partition \((z, w=1, \ldots, q_j)\):

\[
G_{j, wz}^2 = \frac{\sum_{i=1}^{n_z} \sum_{r=1}^{n_w} |x_{iw} - x_{iz}|}{(\mu_z + \mu_w) n_z n_w}.
\]

(18)

**PROPOSITION 1.** When the population \(P\) is partitioned in a first partition \(K\), for which the groups are in turn partitioned along a second partition \(S\), we obtain a multilevel Gini decomposition such as:

\[
G = \sum_{j=1}^{k} \left( \sum_{z=1}^{q_j} G_{j,zz}^2 p_{j,zz} s_{j,zz}^2 \right) p_j s_j + \sum_{j=1}^{k} \left( \sum_{z=2}^{q_j} \sum_{w=1}^{z-1} G_{j,wz}^2 \left( p_{j,ws} s_{j,ws}^2 + p_{j,z} s_{j,z}^2 \right) \right) p_j s_j + \sum_{j=1}^{k} \sum_{h=1}^{k} G_{jh} \left( p_j s_h + p_h s_j \right).
\]

(19)

where the population share and the income share of the \(z\)-th sub-group are:

\[
p_{j,zz}^2 = \frac{n_z}{n_j} ; \quad s_{j,zz}^2 = \frac{n_z \mu_z}{n_j \mu_j}.
\]

(20)

(i) \(G^{w,S}\) is the contribution of the inequalities along the second partition \(S\) within \(K\);

(ii) \(G^{gb,SK}\) is the gross contribution of the inequalities between the sub-populations of the second partition \(S\) within the first partition \(K\) (the gross second-order between-group inequality along \(S\) within \(K\));

(iii) \(G^{gb,K}\) is the gross contribution of the between-group inequality along the first partition \(K\) (the gross first-order between-group inequality along \(K\)).

**PROOF.** The Gini decomposition in two elements proposed by Dagum (1997a, 1997b) is:
This holds for the first partition $K$. It is also holds for the second partition $S$ within $P_j$:

$$G_j = \sum_{k=1}^{\alpha-1} G_{j,k} p_j s_j + \sum_{k=2h=1}^{\alpha-1} G_{j,2h} \left( p_j s_h + p_h s_j \right).$$

(21)

COROLLARY 1. The multilevel Gini decomposition computed on two partitions $K$ and $S$, can be generalized to a finite $\alpha$ number of partitions.

PROOF. Let $\alpha$ be the number of partitions. $^3$ Let $G^{\alpha}_{j,zz}$ be the Gini ratio associated with the $z$-th sub-group of the $\alpha$-th partition (issued from the $j$-th group of the first partition) and $G^{\alpha}_{j,wz}$ be the Gini ratio associated with the $w$-th and the $z$-th sub-groups of the $\alpha$-th partition. Let $p_{j,z}^{\alpha}$ and $s_{j,z}^{\alpha}$ be the population share and the income share of the $z$-th sub-group of the $\alpha$-th partition within the $j$-th first partition. Let $q_j^{\alpha}$ be the number of groups within the $\alpha$-th partition issued from group $P_j$ of the first partition. Thus, equation (19) can be rewritten as:

$$G = \sum_{j=1}^{k} \left\{ \sum_{z=1}^{q_j^{\alpha}} \left( \sum_{z=1}^{q_j^{\alpha}} G^{\alpha}_{j,zz} p_{j,z}^{\alpha} s_{j,z}^{\alpha} \right) \cdots p_{j,z}^{\alpha} s_{j,z}^{\alpha} \right\} p_j s_j +$$

$$+ \sum_{j=2h=1}^{k} \sum_{z=1}^{q_j^{\alpha}} G_{j,2h} \left( p_j s_h + p_h s_j \right) +$$

(23)

$^2$ When the Gini decomposition (8) is directly implemented on the second partition, it is possible to find a standard decomposition in two components, that are: (i) the contribution of the inequalities along the second partition $S$ within the overall population $P \left[ G^{w,S} \right]$; (ii) and the gross between-group inequalities along the partition $S$ within $P \left[ G^{wh,S} \right]$. That means that the overall inequality is defined with Gini coefficients related to the last partition.

$^3$ If $\alpha = 1$, we have the sole partition $K$. If $\alpha = 2$, we have the second partition $S$ within $K$. 
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\[ + \sum_{j=1}^{k} \left( \sum_{z=2}^{q-1} \sum_{w=1}^{r} G_{j,wz}^2 \alpha_{j,wz}^2 P_{j,z}^2 s_{j,z}^2 \right) P_{j} s_{j} + (G_{gb,2K}^\alpha) \]

\[ \vdots \]

\[ + \sum_{j=1}^{k} \left( \sum_{z=2}^{q} \sum_{w=1}^{r} \left( \sum_{j=2}^{q} \sum_{z=2}^{q} G_{j,wz}^\alpha \alpha_{j,wz}^\alpha P_{j,z}^\alpha s_{j,z}^\alpha \right) \right) P_{j} s_{j} \times (G_{gb,\alpha K}^\alpha) \]

where the weights associated with the Gini index between the \( w \)-th and the \( z \)-th groups of the \( \alpha \)-th partition are:

\[ \alpha_{j,wz}^\alpha = P_{j,w} s_{j,z} + P_{j,z} s_{j,w} \]  

Expression \( G_{w,\alpha}^\alpha \) stands for the within-group inequalities along the \( \alpha \)-th partition within \( K \) and \( G_{gb,K} \) is the gross first-order between-group inequality along \( K \). \( G_{gb,2K}^\alpha \) is the gross second-order between-group inequality along the second partition \( S \) within \( K \), which is equivalent to \( G_{gb,SK}^\alpha \) in (19). \( G_{gb,\alpha K}^\alpha \) is the gross \( \alpha \)-order between-group inequality along the \( \alpha \)-th partition within \( K \). Consequently, we obtain the generalization of the three-term Gini decomposition (15) in a multilevel context.

The within-group Gini indices \([G_{w}^\alpha \text{ and } G_{w,\alpha}^\alpha]\) converge towards 0 when the number of groups is equal to the number of individuals. The between-group Gini indices \([G_{gb,K}^\alpha \text{ and } G_{gb,\alpha K}^\alpha]\) converge towards 0 when the number of groups tends towards one. Proposition 1 relies on the gross between-group inequalities. A five-term decomposition can be implemented adding the role of Gini’s transvariation. We include the economic distance between the \( z \)-th and the \( w \)-th distributions:

\[ D_{j,wz}^2 = \frac{d_{wz} - p_{wz}}{d_{wz} + p_{wz}} \]  

**PROPOSITION 2.** When the overall population \( P \) is separated in two partitions (the second partition \( S \) along the first partition \( K \)), there exists a multilevel Gini decomposition in five components, which brings out the intensity of transvariation between the groups and the sub-groups of the first and the second partition respectively:
\[
G = \sum_{j=1}^{k} \left( \sum_{z=1}^{q_j} G_{j,z}^{2} p_{j,z} s_{j,z}^2 \right) p_j s_j + \quad (G^{w,S}) \quad (26)
\]

\[
+ \sum_{j=1}^{k} \left( \sum_{z=2}^{q_j} G_{j,z}^{2} \left( p_{j,z} s_{j,w}^2 + p_{j,w} s_{j,z}^2 \right) D_{j,w}^2 \right) p_j s_j + \quad (G^{nb,SK})
\]

\[
+ \sum_{j=1}^{k} \left( \sum_{z=2}^{q_j} G_{j,z}^{2} \left( p_{j,z} s_{j,w}^2 + p_{j,w} s_{j,z}^2 \right) (1 - D_{j,w}^2) \right) p_j s_j + \quad (G^{SK})
\]

\[
+ \sum_{j=2h=1}^{k} \sum_{j=2h+1}^{k} G_{jh} \left( p_j s_h + p_h s_j \right) D_{jh} + \quad (G^{nb,K})
\]

\[
+ \sum_{j=2h=1}^{k} \sum_{j=2h+1}^{k} G_{jh} \left( p_j s_h + p_h s_j \right) (1 - D_{jh}) \right) . \quad (G^{K})
\]

(i) \(G^{w,S}\) is the contribution of the inequalities along the second partition \(S\) within the first partition \(K\);  
(ii) \(G^{nb,SK}\) is the net contribution of the inequalities between the sub-populations of the sub-partition \(S\) within the partition \(K\) (the net second-order between-group inequality);  
(iii) \(G^{SK}\) is the intensity of transvariation between the sub-populations of the sub-partition \(S\) within the partition \(K\) (the second-order between-group inequality of transvariation along \(S\) within \(K\));  
(iv) \(G^{nb,K}\) is the net contribution of the inequality between the groups along the first partition \(K\) (the net first-order between-group inequality);  
(v) \(G^{K}\) is the intensity of transvariation between the groups along the first partition \(K\) (the first-order between-group inequality of transvariation).

**Proof.** First, multiply \(G^{nb,SK}\) by the economic distance \([D_{j,w}^2]\) and by the ratio of overlap \([1 - D_{j,w}^2]\). Second, multiply \(G^{nb,K}\) by the economic distance \([D_h]\) and by the ratio of overlap \([1 - D_{jh}]. \)**

**Corollary 2.** From the five-term multilevel Gini decomposition (26) computed on two partitions \(K\) and \(S\), a generalization to \(\alpha\) partitions is deduced.  
**Proof.** See Appendix 1. **

The multilevel Gini decompositions explain the sub-group determinants of the overall inequality, and are of interest when we aim at analysing both inequality and poverty.
3.2 A poor–non-poor inequality analysis

Suppose we have only two sub-populations (poor and non-poor) within each group of the first partition $K$. Consequently, we obtain a particular case of proposition 2, since the income distributions of the poor and the non-poor sub-groups do not overlap [$D_{j,wz}^2 = 1$].

**COROLLARY 3.** When each group $P_j$ of the first partition $K$ is partitioned in two non-overlapping sub-groups such as poor and non-poor sub-populations, the multilevel Gini decomposition in five elements becomes a four-element equation as:

\[
G = \frac{1}{2} \left( \sum_{j=1}^{k} \left( \sum_{z=1}^{2} G_{j,z}^2 p_{j,z}^2 s_{j,z}^2 \right) p_j s_j \right) + \sum_{j=1}^{k} \left( G_{w,S}^j \right) (27)
\]

As $D_{j,wz}^2 = 1$, we immediately have:

(i) $G_{w,S}$ is the contribution of the inequalities along the poor and the non-poor sub-populations within the first partition $K$;

(ii) $G_{nh,SK}^j$ is the net contribution of the inequalities between the poor and the non-poor sub-populations within the partition $K$;

(iii) $G_{nh,K}^j$ is the net contribution of the inequalities between the groups of the first partition $K$;

(iv) $G_{jk}$ is the intensity of transvariation between the groups of $K$.

**PROOF.** The number of sub-populations within each group $P_j$ of the first partition is: $q_j^2 = 2$. Thus, $G_{w,S}$ (the contribution of the within-group inequalities along the second partition $S$ of the first partition $K$) can be reformulated as:

\[
G = \frac{1}{2} \left( \sum_{j=1}^{k} \left( \sum_{z=1}^{2} G_{j,z}^2 p_{j,z}^2 s_{j,z}^2 \right) p_j s_j \right) + \sum_{j=1}^{k} \left( \sum_{z=1}^{2} G_{j,z}^2 p_{j,z}^2 s_{j,z}^2 \right) p_j s_j . \quad (28)
\]

As $D_{j,wz}^2 = 1$, we immediately have:
\[ G_{j}^{SK} = \sum_{j=1}^{k} \left( G_{j,we}^{2} \left( p_{j,z}^{2} s_{j,w}^{2} + p_{j,w}^{2} s_{j,z}^{2} \right) \left( 1 - D_{j,we}^{2} \right) \right) p_{j} s_{j} = 0 . \quad (29) \]

On the other hand, \( G_{nb,SK} \) (the net second-order between-group inequality) is:

\[ G_{nb,SK} = \sum_{j=1}^{k} \left( \sum_{i=1}^{2} \sum_{z=1}^{2} G_{j,we}^{2} \left( p_{j,z}^{2} s_{j,w}^{2} + p_{j,w}^{2} s_{j,z}^{2} \right) D_{j,we}^{2} \right) p_{j} s_{j} + \quad (30) \]

\[ = \sum_{j=1}^{k} \left( G_{j,we} \left( p_{j,z}^{2} s_{j,w}^{2} + p_{j,w}^{2} s_{j,z}^{2} \right) \right) p_{j} s_{j} . \quad * \]

**EXAMPLE OF INTERPRETATION.** Imagine a spatial decomposition. The first partition is a regional one, whereas the second one deals with poor/non-poor income distributions, after introducing an adequate poverty line. In this case, applying the multilevel Gini decomposition yields four indices: (i) \( G_{w,S} \) measures the inequalities of the poor and the non-poor sub-populations within each region; (ii) \( G_{nb,SK} \) gives the net inequalities between the poor and the non-poor sub-populations along each region; (iii) \( G_{nb,K} \) gauges the net between-group inequalities between each and every pair of regions; and (iv) \( G_{t,K} \) estimates the intensity of transvariation between regions.

Finally, this kind of multilevel decomposition is useful to analyse poor and non-poor distributions and particularly, their impacts on the overall inequality.

4. The multi-decomposition

Mussard (2004, 2006) provided a transparent link between the Gini decomposition by sub-group and the Gini decomposition by factor component in introducing the income source decomposition into the Gini decomposition by sub-group. Let us present the income source method with the well-known formula:

\[ \left| x_{i} - x_{r} \right| = x_{i} + x_{r} - 2 (x_{i} \land x_{r}) \quad (31) \]

where \((x_{i} \land x_{r})\) selects the minimum value between the incomes \(x_{i}\) and \(x_{r}\).

The Gini ratio can be expressed as follows:

\(^{4}\) This is also feasible for the Gini mean difference and the Gini mean ratio [see Mussard (2004)].
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\[ G = \frac{\sum_{i=1}^{n} \sum_{r=1}^{n} (x_i + x_r - 2(x_i \wedge x_r))}{2\mu_{nn}}. \]  

(32)

The incomes are aggregated along \( q \) income sources (such as labour income, capital income, child support benefits, taxes, etc.). Then, the \( i \)-th individual’s income is:

\[ x_i = \sum_{m=1}^{q} x_{im}. \]

(33)

The overall Gini ratio is now rewritten as:

\[ G = \frac{\sum_{i=1}^{n} \sum_{r=1}^{n} \left( \sum_{m=1}^{q} x_{im}^m + \sum_{m=1}^{q} x_{rm}^m - 2(x_i \wedge x_r) \right)}{2\mu_{nn}}. \]

(34)

We define the income source decomposition of \( 2(x_i \wedge x_r) \) by:

\[ \sum_{m=1}^{q} 2x_{im}^m := 2(x_i \wedge x_r), \]

(35)

where \( x_{im}^m \) is the \( m \)-th source of the minimum value between incomes \( x_i \) and \( x_r \).

**PROPOSITION 3.** Given (34) and (35) a Gini decomposition by factor component can be expressed as:

\[ G = \sum_{m=1}^{q} \left( \frac{\sum_{i=1}^{n} \sum_{r=1}^{n} (x_{im}^m + x_{rm}^m - 2x_{im}^m)}{2\mu_{nn}} \right) = \sum_{m=1}^{q} (s^m), \]

(36)

where \( s^m \) is the contribution of the \( m \)-th source to \( G \).

**PROOF.** It is straightforward. ◆

**COROLLARY 4.** Let \( x^m \) be the \( n \times 1 \) vector the \( m \)-th income source. If \( x^1 = x^2 = \ldots = x^m \), we then have:
\[ S^1 = \ldots = S^m = \ldots = S^n \Rightarrow G = q S^m. \] (37)

**Proof.** It is straightforward. ♦

Corollary 4 indicates that the source contributions are proportional to the Gini index. This property is important since the rules of source decomposition are diverging on this point [see e.g. Shorrocks’ Shapley decompositions (1999), which are not proportional]. Consequently, if the factor distributions are equally distributed, then the source contributions are equal to zero.

It is worth mentioning that, in many cases, the contribution \( S^m \) may be negative, implying that a factor component such as transfers or pensions decrease the overall inequality. This is coherent with policies of redistribution that intend to alleviate global inequality.\(^5\)

As the Gini index associated with one group \( [G_{jj}] \), or with two groups \( [G_{jh}] \) has the same structure than the overall Gini ratio, these indices are factor decomposable:

\[
G_{jj} = \sum_{m=1}^{q} \left\{ \sum_{i,j} \left( x_{ij}^m + x_{rj}^m - 2 x_{ijrj}^m \right) \right\} \frac{2 \mu_j n_j n_j}{\sum_{i,j} x_{ij}} \sum_{m=1}^{q} \left( S_{jj}^m \right), \tag{38}
\]

\[
G_{jh} = \sum_{m=1}^{q} \left\{ \sum_{i,k} \left( x_{ij}^m + x_{rj}^m - 2 x_{ijrj}^m \right) \right\} \frac{2 \mu_j \mu_k n_j n_h}{\sum_{i,k} x_{ij}} \sum_{m=1}^{q} \left( S_{jh}^m \right), \tag{39}
\]

where \( x_{ijrj}^m \) is the \( m \)-th source of the minimum value between \( x_{ij} \) and \( x_{rj} \), and where \( x_{ijrj}^m \) is the \( m \)-th source of the minimum value between \( x_{ij} \) and \( x_{rj} \).

Therefore, the introduction of these source decompositions into the subgroup decomposition yields a synthetic decomposition, the so-called multi-decomposition.

**Proposition 4 [Mussard (2004, 2006)].** The Gini index is two-term multi-decomposable within an exact structure.

\(^5\) For example, the variance and the coefficient of variation always lead to positive contributions. Then, it is more difficult to appreciate the incidence of inequality-reducing tax reforms.
PROOF. We substitute (38) and (39) in Dagum’s Gini decomposition in two elements (8):

\[ G = \sum_{j=1}^{k} \sum_{m=1}^{q} \left( S_{jj}^m \right) p_j s_j + \] (Gw) (40)

\[ + \sum_{j=2}^{k} \sum_{h=1}^{k-1} \sum_{m=1}^{q} S_{jh}^m \left( p_j s_h + p_h s_j \right). \] (Gwb)

This decomposition in two dimensions entails the estimation of the combinations “group j / source m” and “between groups j and h / source m” without redundant terms (except the denominator). The multi-decomposition is exact:

\[ G = \sum_{m=1}^{q} \left( \frac{1}{2} \sum_{j=1}^{k} \sum_{i=1}^{n_j} \sum_{r=1}^{n_i} (x_{ij}^m + x_{ir}^m - 2x_{ijr}^m) \right) \left( \frac{1}{2 \mu_{nn}} \right) + \sum_{m=1}^{q} \left( 2 \sum_{j=2}^{k} \sum_{h=1}^{j-1} \sum_{i=1}^{n_j} \sum_{r=1}^{n_i} (x_{ij}^m + x_{ir}^m - 2x_{ihr}^m) \right) \left( \frac{1}{2 \mu_{nn}} \right). \] (41)

**Proposition 5 [Mussard (2004, 2006)].** The Gini index is three-term multi-decomposable within an exact structure.

PROOF. Firstly, decompose the economic distance between the j-th and the h-th groups by factor component:

\[ D_{jh} = \sum_{m=1}^{q} D_{jh}^m, \forall j > h \] (42)

where,

\[ D_{jh}^m = \left( \frac{1}{\sum_{i=1}^{n_j} \sum_{r=1}^{n_i} (x_{ij}^m + x_{ir}^m - 2x_{ihr}^m) - \sum_{i=1}^{n_j} \sum_{r=1}^{n_i} (x_{ij}^m + x_{ir}^m - 2x_{ihr}^m)} \right). \] (43)

Secondly, decompose the (1–D_{jh}) statistics, say the overlap ratio P_{jh}. 
$1 - D_{jh} = P_{jh} = \sum_{m=1}^{g} \frac{2}{\sum_{i,j}^{n_j} x_{ij} + x_{rh} - 2x_{ijh}} \sum_{j=1}^{q_n} \sum_{h=1}^{q_m} |x_{ij} - x_{rh}| = \sum_{m=1}^{g} P_{mh}, \forall \mu_j > \mu_h, \quad (44)$

where $P_{mh}$ is the contribution of the $m$-th source to $P_{jh}$. This overlap ratio tends towards one when the $j$-th and the $h$-th distributions completely overlap, and tends towards zero when the distributions do not overlap. Hence, substituting (43) and (44) in the sub-group decomposition (15) gives:

$$G = \sum_{j=1}^{k} \sum_{m=1}^{q} \left( \sum_{i,j}^{n_j} x_{ij} + x_{rh} - 2x_{ijh} \right) + \frac{2\mu_{nn}}{(G_w)} \left( \sum_{j=1}^{k} \sum_{i,j}^{n_j} x_{ij} + x_{rh} - 2x_{ijh} \right) \quad (G_u) \quad (45)$$

$$+ \sum_{j=2h=1}^{k} \sum_{m=1}^{q} D_{mh} \left( p_j s_h + p_h s_j \right) G_{jh} + \quad (G_{ub})$$

$$+ \sum_{j=2h=1}^{k} \sum_{m=1}^{q} P_{mh} \left( p_j s_h + p_h s_j \right) G_{jh} . \quad (G_i)$$

This three-term multi-decomposition is exact since there are no redundant terms:

$$G = \sum_{m=1}^{q} \left\{ \frac{2\mu_{nn}}{(G_w)} \left( \sum_{j=1}^{k} \sum_{i,j}^{n_j} x_{ij} + x_{rh} - 2x_{ijh} \right) \right\} + \quad (G_u) \quad (46)$$

$$+ \sum_{j=2h=1}^{k} \sum_{m=1}^{q} \left\{ \frac{2\mu_{nn}}{(G_{ub})} \left( \sum_{i,j}^{n_j} x_{ij} + x_{rh} - 2x_{ijh} \right) \right\}$$

$$+ \sum_{m=1}^{q} \left\{ \frac{4\mu_{nn}}{(G_i)} \left( \sum_{i,j}^{n_j} x_{ij} + x_{rh} - 2x_{ijh} \right) \right\} . \quad (G_i)$$

It is possible to compute exactly the contribution of each income source to the within-group Gini index $[G_w]$, to the net between-group Gini index $[G_{ub}]$, and to the intensity of transvariation $[G_i]$. Precisely, we have: (i) the contribution of the $m$-th income source of the $j$-th group to the total Gini ratio; (ii) the contribution of the $m$-th income source of the net inequalities
between the \(j\)-th and the \(h\)-th groups to \(G\); and (iii) the contribution of the \(m\)-th income source of transvariation between the \(j\)-th and the \(h\)-th groups to \(G\).

**COROLLARY 5.** The multi-decomposition and the multilevel Gini decomposition for two partitions (26) are combined such as:

\[
G = \sum_{j=1}^{k} \left( \sum_{z=1}^{q} \sum_{m=1}^{q} S_{j,z}^{2,m} P_{j,z}^{2} S_{j,z}^{2} \right) P_{j,z} \quad \text{(G}_{n}.S) (47)
\]

\[
+ \sum_{j=1}^{k} \left( \sum_{z=2}^{q} \sum_{m=1}^{q} D_{j,wz}^{2,m} G_{j,wz}^{2} \left( P_{j,z}^{2} S_{j,w}^{2} + P_{j,w}^{2} S_{j,z}^{2} \right) \right) P_{j,z} \quad \text{(G}_{n}.S\text{K})
\]

\[
+ \sum_{j=1}^{k} \left( \sum_{z=2}^{q} \sum_{m=1}^{q} P_{j,wz}^{2,m} G_{j,wz}^{2} \left( P_{j,z}^{2} S_{j,w}^{2} + P_{j,w}^{2} S_{j,z}^{2} \right) \right) P_{j,z} \quad \text{(G}_{n}.\text{SK})
\]

\[
+ \sum_{j=2b-1}^{k} \sum_{m=1}^{q} D_{j,h}^{m} G_{j,h} \left( P_{j,h} S_{j} + P_{h} S_{j} \right) \quad \text{(G}_{n}.\text{h.K})
\]

\[
+ \sum_{j=2b-1}^{k} \sum_{m=1}^{q} P_{j,h}^{m} G_{j,h} \left( P_{j,h} S_{j} + P_{h} S_{j} \right) . \quad \text{(G}_{n}.\text{h.K})
\]

**PROOF.** On the one hand, we decompose the Gini coefficient of the \(z\)-th sub-group of group \(P_{j}\) of the first partition \(G_{j,zz}^{2}\) by income source. We obtain the contribution of each source \(S_{j,zz}^{2,m}\) to \(G_{j,zz}^{2}\). On the other hand, we decompose the economic distance between the groups of the first and the second partition (\(D_{j,wz}^{2,m}\) and \(D_{j,h}^{m}\) respectively) in order to obtain a mixture of decomposition that relies on multilevel and multi-decomposition techniques. Then, all the sub-group determinants of the overall inequality are explained by each income source and could be generalised to \(\alpha\) partitions.

**5. Econometric modelisation**

In the preceding section, the inequalities are determined by groups and sources. Analysing only income sources limits considerably the number of dimensions, say attributes. To extend the multi-decomposition in a multidimensional context, we linearly model incomes by \(q\) explanatory variables [see Morduch and Sicular (2002)].
PROPOSITION 6. When incomes \((x_i)\) are linearly regressed on \(q\) socio-economic variables \(X\)’s (for example economic conditions and social policy indicators: health, education level, etc.) as follows:

\[
x_i = \sum_{m=0}^{q} a_m X_i^m + \varepsilon_i,
\]

where \(X_i^0=1\) and \(\varepsilon_i\) is the random error term, then \(G_w\), \(G_{nb}\), and \(G_t\) are determined by the \(q\) attributes:

\[
G = \sum_{j=1}^{k} \sum_{m=0}^{q+1} (S_j^m) p_j s_j + \quad (G^w) \quad (49)
\]

\[
+ \sum_{j=2}^{k} \sum_{h=1}^{q} \sum_{m=0}^{q+1} D_{jh}^m \left( p_j s_h + p_h s_j \right) G_{jh} + \quad (G^{nb})
\]

\[
+ \sum_{j=2}^{k} \sum_{h=1}^{q} \sum_{m=0}^{q+1} P_{jh}^m \left( p_j s_h + p_h s_j \right) G_{jh} . \quad (G^t)
\]

PROOF. All \(x_i\)’s are linearly disaggregated. Indeed :

\[
x_i = x_i^0 + x_i^1 + \ldots + x_i^q + x_i^{q+1},
\]

where \(x_i^0\) represents the intercept, \(x_i^1\) is dimension 1, \(x_i^q\) is dimension \(q\), and \(x_i^{q+1}\) is the estimated residual \(\hat{\varepsilon}_i\). Then, we find exactly the same result as in the non-parametric case except we sum over \(q+2\) sources.

This enlarges considerably the multi-decomposition (46) since the \(q\) variables are not standard income sources. Therefore, using the t-Student tests, we isolate the significant variables that tend to increase the overall inequality.

COROLLARY 6. Equation (49) can be merged with the multilevel decomposition for two partitions (26) such as:

\[
G = \sum_{j=1}^{k} \left( \sum_{z=1}^{q} \sum_{m=0}^{q+1} S_{j,z}^m p_j s_j^2 \right) p_j s_j + \quad (G^{w,S})
\]

\[6\] Indeed, we have \(q\) dimensions plus the contribution of the intercept and that of the error term. Note also that a necessary and sufficient condition to obtain a multi-decomposition is the linearity of the econometric model. In consequence, complicated linear models may be introduced in the Gini index multi-decomposition.
EXTENSIONS OF DAGUM’S GINI DECOMPOSITION

\[
\begin{align*}
&+ \sum_{j=1}^{k} \left( \sum_{z=2}^{q-1} \sum_{j, w=0}^{m} D_{j,w}^{z} G_{j,wz}^{z} \left( p_{j,w}^{2} s_{j,z}^{2} + p_{j,z}^{2} s_{j,w}^{2} \right) \right) p_{j} s_{j} + (G_{nb,SK}) \\
&+ \sum_{j=1}^{k} \left( \sum_{z=2}^{q-1} \sum_{j, w=0}^{m} P_{j,w}^{z} G_{j,wz}^{z} \left( p_{j,w}^{2} s_{j,z}^{2} + p_{j,z}^{2} s_{j,w}^{2} \right) \right) p_{j} s_{j} + (G_{b,SK}) \\
&+ \sum_{j=2}^{k} G_{j,h}^{m} \left( p_{j} s_{h} + p_{h} s_{j} \right) + (G_{nb,K}) \\
&+ \sum_{j=2}^{k} G_{j,h}^{m} \left( p_{j} s_{h} + p_{h} s_{j} \right) . (G_{b,K})
\end{align*}
\]

PROOF. It is straightforward. ♦

The practical difficulty of this model relies on the quality of adjustment. If the overall variance is enough explained by the regression model, one can logically expect that the different attributes make clear the evolution of the inequalities within groups and between groups (of particular partitions).7

6. A case of study: Luxembourg

This study deals with the multi-decomposition and the multi-level Gini index decomposition [equation (47)]. This method is applied to Luxembourg. The database used in this study comes from the survey Socio-Economic Panel Lieven zu Lëtzebuerg (PSELL-2) which includes information about life conditions and market characteristics in Luxembourg obtained via the « Consortium of Household Panels for European Socio-Economic Research » (CHER). This survey has been performed every year since 1994 beside a representative survey of households and the individuals living in Luxembourg. The application covers households in 2001.

7 Econometric models may lead to approximations, at different steps, which summing together, could not guarantee a good representation of the data and thus they could conduct to some risky conclusions. It is then necessary to obtain low contributions to the overall Gini index for the intercept and for the error term.
Table 1: Studied groups and sub-groups of population

<table>
<thead>
<tr>
<th>First Partition</th>
<th>$p_j$</th>
<th>$s_j$</th>
<th>Second Partition</th>
<th>$p_{j,z}$</th>
<th>$s_{j,z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Urban: $j=1$</td>
<td>67.76</td>
<td>65.90</td>
<td>Nat Urban: $z=1$</td>
<td>65.02</td>
<td>65.90</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Not Nat Urban: $z=2$</td>
<td>34.98</td>
<td>34.10</td>
</tr>
<tr>
<td>Rural: $j=2$</td>
<td>32.24</td>
<td>34.10</td>
<td>Nat Rural: $z=1$</td>
<td>72.57</td>
<td>73.69</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Not Nat Rural: $z=2$</td>
<td>27.43</td>
<td>26.31</td>
</tr>
</tbody>
</table>

$p_j$: percent of total population belonging to each group of the first partition
$s_j$: percent of global income posses by each group
$p_{j,z}$: percent of total population belonging to each sub-group of the second partition
$s_{j,z}$: percent of global income posses by each sub-group within each group

In order to illustrate the multi-level Gini decomposition, the households are grouped according to a first partition: people living in town (Group 1: Urban) which represents almost 68% of total population and possess 66% of global income; and people living in rural areas (Group 2: Rural), that is 32% of the population having only 34% of global income (See Table 1). The population is decomposed into a second partition: households with Luxemburger nationality (Sub-group 1: Nat Urban); households living in town with foreign nationality (Sub-group 2: Not Nat Urban); households living in rural areas with Luxemburger nationality (Sub-group 1: Nat Rural); and households living in rural areas with a foreign nationality (Sub-group 2: Not Nat Rural).

The Gini indices for the four sub-groups give the following information: there are strong income inequalities between households living in rural areas of Luxemburger nationality ($G_{2,11}^2 = 0.312$). Followed by inequalities concerning households living in town with Luxemburger nationality ($G_{2,11}^2 = 0.307$). Then, we find foreign households living in town ($G_{2,22}^2 = 0.304$) and those living in rural areas ($G_{2,22}^2 = 0.289$). These indicators provide important information. Nevertheless, they do not allow us to know exactly neither the contribution of each sub-group to the level of overall inequality nor their interaction. The global Gini index is $G = 0.307$. We propose to decompose this index using the multidimensional technique in order to study the contributions of each sub-group of population and each income sources to the overall inequality. The selected income sources are: salaries, incomes from trade and income of artisans, rents, private income transfers, dismissal’s indemnity, transfers due to handicaps and health, family transfers, other transfers, pensions, and other income sources.

Tables 2, 3 and 4 present the results from the multi-decomposition of the Gini index.8 Tables 2a and 2b show the determinants of within-group

---

8 Tables yield relative contributions in percentage of the overall Gini index. For example, to calculate the within group inequalities for the not natural urban group and for salaries, we have divided the absolute value of $G_{1,22}^{0.0142}$ to the global
inequalities. We can notice that the sub-group $j = 1, z = 1$ (Nat Urban) has the biggest contribution to global inequality (explaining 19.14% of $G$). The main sources explaining these inequalities are salaries (13.68%) and rents (2.61%). Contributions can be negatives, meaning that these income sources contribute to reduce global inequality. The Nat Urban sub-group also decreases global inequalities thanks to private income transfers (-0.03%), dismissal’s indemnity (-0.07) and pensions (-0.55).

Table 2a: Within-group contributions to global inequality

<table>
<thead>
<tr>
<th></th>
<th>Salaries</th>
<th>Incomes from artisans and trade</th>
<th>Rents</th>
<th>Private transfers</th>
<th>Dismissals's indemnity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{w1,11}$</td>
<td>13.68</td>
<td>1.34</td>
<td>2.61</td>
<td>-0.03</td>
<td>-0.07</td>
</tr>
<tr>
<td>$G_{w1,22}$</td>
<td>4.63</td>
<td>0.26</td>
<td>0.20</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$G_{w2,11}$</td>
<td>3.84</td>
<td>0.52</td>
<td>0.36</td>
<td>0.00</td>
<td>-0.07</td>
</tr>
<tr>
<td>$G_{w2,22}$</td>
<td>0.62</td>
<td>0.07</td>
<td>0.03</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 2b: Within-group contributions to global inequality

<table>
<thead>
<tr>
<th></th>
<th>Transfers due to handicaps and health</th>
<th>Family transfers</th>
<th>Other transfers</th>
<th>Pensions</th>
<th>Other income sources</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{w1,11}$</td>
<td>0.07</td>
<td>1.99</td>
<td>0.10</td>
<td>-0.55</td>
<td>0.00</td>
</tr>
<tr>
<td>$G_{w1,22}$</td>
<td>0.03</td>
<td>0.36</td>
<td>0.03</td>
<td>-0.23</td>
<td>0.00</td>
</tr>
<tr>
<td>$G_{w2,11}$</td>
<td>0.03</td>
<td>1.17</td>
<td>0.10</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$G_{w2,22}$</td>
<td>0.00</td>
<td>0.03</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Between-group inequalities are exposed in Tables 3a and 3b. The most important contribution to inequalities are those between Luxemburger ($j = 1, z = 1$) and foreign households living in town ($j = 1, z = 2$). The main sources explaining these contributions are salaries and incomes ($G_{nb1,12}$: incomes from artisans and trade). It is also between households belonging to these groups that transfers and pensions can reduce inequalities.

Table 3a: Between-group contributions to global inequality

<table>
<thead>
<tr>
<th></th>
<th>Salaries</th>
<th>Incomes from artisans and trade</th>
<th>Rents</th>
<th>Private transfers</th>
<th>Dismissals's indemnity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{nb1,12}$</td>
<td>27.23</td>
<td>2.35</td>
<td>4.10</td>
<td>-0.10</td>
<td>-0.52</td>
</tr>
<tr>
<td>$G_{nb2,12}$</td>
<td>22.35</td>
<td>1.14</td>
<td>0.29</td>
<td>-0.10</td>
<td>-0.07</td>
</tr>
<tr>
<td>$G_{nb1,22}$</td>
<td>4.10</td>
<td>0.29</td>
<td>0.03</td>
<td>-0.03</td>
<td>-0.03</td>
</tr>
</tbody>
</table>

Gini index value (0.0307) multiply by 100. This result means that 4.63% of the overall Gini index is explained by the salaries of group $j=1$ and $z=2$ (Not Nat Urban).
### Table 3b: Between-group contributions to global inequality

<table>
<thead>
<tr>
<th></th>
<th>Transfers due to handicaps and health</th>
<th>Family transfers</th>
<th>Other transfers</th>
<th>Pensions</th>
<th>Other income sources</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{m_{1,12}}^{G_{1}}$</td>
<td>0.26</td>
<td>2.80</td>
<td>0.13</td>
<td>2.21</td>
<td>0.00</td>
</tr>
<tr>
<td>$G_{m_{2,12}}^{G_{2}}$</td>
<td>-0.23</td>
<td>2.25</td>
<td>0.10</td>
<td>-6.84</td>
<td>0.00</td>
</tr>
<tr>
<td>$G_{m_{3,12}}^{G_{3}}$</td>
<td>-0.03</td>
<td>0.46</td>
<td>0.03</td>
<td>-1.04</td>
<td>0.00</td>
</tr>
</tbody>
</table>

These Tables present the details for different types of transfers and their role in global inequalities. Some contributions are negatives. For example, transfers for ill persons (or handicap persons) allow for reducing inequalities between residents and non residents, living either in urban or rural areas. In the same way, pensions reduce inequalities between sub-groups. Contrary to this, transfers associated to family do not allow to reduce the between contributions to inequalities.

### Table 4a: Between-groups contributions of inequalities of transvariation to global inequality

<table>
<thead>
<tr>
<th></th>
<th>Salaries</th>
<th>Incomes from artisans and trade</th>
<th>Rents</th>
<th>Private transfers</th>
<th>Dismissals’s indemnity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{t_{1,12}}^{T_{1}}$</td>
<td>5.37</td>
<td>1.11</td>
<td>-0.65</td>
<td>0.00</td>
<td>0.26</td>
</tr>
<tr>
<td>$G_{t_{2,12}}^{T_{2}}$</td>
<td>-6.58</td>
<td>0.10</td>
<td>1.50</td>
<td>0.07</td>
<td>0.03</td>
</tr>
<tr>
<td>$G_{t_{3,12}}^{T_{3}}$</td>
<td>-1.01</td>
<td>0.07</td>
<td>0.16</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

### Table 4b: Between-groups contributions of inequalities of transvariation to global inequality

<table>
<thead>
<tr>
<th></th>
<th>Transfers due to handicaps and health</th>
<th>Family transfers</th>
<th>Other transfers</th>
<th>Pensions</th>
<th>Other income sources</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{t_{1,12}}^{T_{1}}$</td>
<td>0.00</td>
<td>2.96</td>
<td>0.33</td>
<td>-3.32</td>
<td>0.00</td>
</tr>
<tr>
<td>$G_{t_{2,12}}^{T_{2}}$</td>
<td>0.36</td>
<td>-0.49</td>
<td>0.00</td>
<td>6.29</td>
<td>0.00</td>
</tr>
<tr>
<td>$G_{t_{3,12}}^{T_{3}}$</td>
<td>0.03</td>
<td>0.10</td>
<td>0.03</td>
<td>1.01</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Finally, the between-group inequalities of transvariation, that is the inequalities explaining that some households from the poorest sub-groups (in mean) (Not Nat Rural) reach to create income disparities with households belonging to richest sub-groups (in mean) (Nat Rural), are sometimes negatives, for example for salaries (see $G_{t_{3,12}}^{T_{3}}$). That means that the poorest sub-groups could not generate income’s disparities with the richest one. For the others income sources, as rents, the transvariation is positive meaning that households belonging to the poorest sub-groups starting to join some levels of income of the richest sub-groups (See Tables 4a and 4b).
7. Conclusions

The Gini index belongs to the class of decomposable measures based on interpersonal comparisons [see Pyatt (1976)]. This principle is crucial since individuals examine perpetually such comparisons. It is not satisfied by the generalized entropy inequality measures [see Dagum (1998)].

Even if the generalized entropy is sub-group decomposable as well as in a multilevel background [see Salas (2002)], it does not fulfil the property of multi-decomposition [see Mussard (2004, 2006)] that merges the sub-group and the income source decompositions.

Furthermore, the Gini decomposition exhibits a third element that is not a residual term since it corresponds to the intensity of transvariation [see Gini (1916) and Dagum (1959, 1960, 1961)], which captures the deprivation of the sub-groups with higher mean incomes. Furthermore, when the overall population is divided into many sub-partitions, the multilevel Gini decomposition offers the intensity of transvariation between the groups of each partition \([G^{\text{SK}}] \text{ and } G^{\text{SK}}\alpha\text{K}]\), the net between-group Gini within each partition \([G^{\text{SK}}] \text{ and } G^{\text{SK}}\alpha\text{K}]\), and the Gini within the groups of each partition \([G^{\text{wS}}]\). Finally, both the multi-decompositions and the parametric multi-decompositions precise, within an exact structure, the components of the overall inequality, which can help decision makers to contemplate socio-economic policies of redistribution.

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REFERENCES


**APPENDIX 1.**

**PROOF OF COROLLARY 2.** We use the same approach as corollary 1:

\[
G = \sum_{j=1}^{k} \left( \sum_{z=2}^{\alpha-1 \text{times}} \left( \sum_{z=2}^{\alpha} G_{j,z}^a p_{j,z}^a s_{j,z}^a \right) \right) p_j s_j + (G^{\alpha,a})
\]

\[
+ \sum_{j=2h+1}^{k} \sum_{j=1}^{\alpha} G_{j,h} \left( p_j s_h + p_h s_j \right) + (G^{\alpha,h,K})
\]

\[
+ \sum_{j=2h}^{k} \sum_{j=1}^{\alpha} G_{j,h} \left( p_j s_h + p_h s_j \right) \left( 1 - D_{j,h} \right) + (G^{K})
\]

\[
+ \sum_{j=1}^{k} \sum_{z=2w=1}^{\alpha-2 \text{ times}} \left( \sum_{z=2}^{\alpha} D_{j,wz}^2 \sigma_{j,wz}^2 \sum_{j=2w=1}^{\alpha-1} \left( \sum_{z=2}^{\alpha} G_{j,wz} D_{j,wz}^2 \sigma_{j,wz}^2 \right) p_{j,z}^a s_{j,z}^a \right) p_j s_j + (G^{\alpha,h,2K})
\]

\[
\vdots
\]

\[
+ \sum_{j=1}^{k} \sum_{z=2w=1}^{\alpha-2 \text{ times}} \left( \sum_{z=2}^{\alpha} D_{j,wz}^2 \sigma_{j,wz}^2 \sum_{j=2w=1}^{\alpha-1} \left( \sum_{z=2}^{\alpha} G_{j,wz} D_{j,wz}^2 \sigma_{j,wz}^2 \right) p_{j,z}^a s_{j,z}^a \right) p_j s_j (G^{\alpha,ak})
\]

\[
+ \sum_{j=1}^{k} \sum_{z=2w=1}^{\alpha-2 \text{ times}} \left( \sum_{z=2}^{\alpha} D_{j,wz}^2 \sigma_{j,wz}^2 \sum_{j=2w=1}^{\alpha-1} \left( \sum_{z=2}^{\alpha} G_{j,wz} D_{j,wz}^2 \sigma_{j,wz}^2 \right) p_{j,z}^a s_{j,z}^a \right) p_j s_j + (G^{\alpha,2K})
\]

\[
\vdots
\]

\[
+ \sum_{j=1}^{k} \sum_{z=2w=1}^{\alpha-2 \text{ times}} \left( \sum_{z=2}^{\alpha} D_{j,wz}^2 \sigma_{j,wz}^2 \sum_{j=2w=1}^{\alpha-1} \left( \sum_{z=2}^{\alpha} G_{j,wz} D_{j,wz}^2 \sigma_{j,wz}^2 \right) p_{j,z}^a s_{j,z}^a \right) p_j s_j (G^{\alpha,ak})
\]

\[D_{j,wz}^2\] is the distance between the \(w\)-th and the \(z\)-th sub-populations of the \(\alpha\)-th partition.